

# Aggregation of autoregressive random fields and anisotropic long memory\*

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## Abstract

We introduce the notion of anisotropic long memory for random fields on  $\mathbb{Z}^2$  whose partial sums on incommensurate rectangles with sides growing at different rates  $O(n)$  and  $O(n^{H_1/H_2})$ ,  $H_1 \neq H_2$  tend to an operator scaling random field on  $\mathbb{R}^2$  with two scaling indices  $H_1, H_2$ . The random fields with such behavior are obtained by aggregating independent copies of a random-coefficient nearest-neighbor autoregressive random fields on  $\mathbb{Z}^2$  with i.i.d. innovations belonging to the domain of attraction of  $\alpha$ -stable law,  $0 < \alpha \leq 2$  with a scalar random coefficient  $A$  (the spectral radius of the corresponding autoregressive operator) having a regularly varying probability density near the ‘unit root’  $A = 1$ . The proofs are based on a study of scaling limits of the corresponding lattice Green functions.

*Keywords:* anisotropic long memory; operator scaling random field; contemporaneous aggregation; autoregressive random field; lattice Green functions;  $\alpha$ -stable mixed moving average

## 1 Introduction

Following Biermé et al. [6], a scalar random field  $\{V(x), x \in \mathbb{R}^d\}$  is called *operator scaling random field* (OSRF) if there exist a  $H > 0$  and a  $d \times d$  real matrix  $E$  whose all eigenvalues have positive real parts, such that for any  $\lambda > 0$

$$\{V(\lambda^E x)\} \stackrel{\text{fdd}}{=} \{\lambda^H V(x)\}. \quad (1.1)$$

(See the end of this section for all unexplained notation.) In the case when  $E = I$  is the unit matrix, (1.1) agrees with the definition of  $H$ -self-similar random field (SSRF), the latter referred to as self-similar process when  $d = 1$ . OSRFs may exhibit strong anisotropy and play an important role in various physical theories, see [6] and the references therein. Several classes of OSRFs were constructed and discussed in [6], [9].

It is well-known that the class of self-similar processes is very large, SSRFs and OSRFs being even more numerous. According to a popular view, the ‘value’ of a concrete self-similar process depends on its ‘domain of attraction’. In the case  $d = 1$ , the domain of attraction of a self-similar stationary increment (sssi) process  $\{V(\tau), \tau \geq 0\}$  is usually defined as the class of all stationary processes  $\{Y(t), t \in \mathbb{Z}_+\}$  whose normalized partial sums tend to  $\{V(\tau)\}$  in the distributional sense, viz.,

$$B_n^{-1} \sum_{t=1}^{[n\tau]} Y(t) \stackrel{\text{fdd}}{\longrightarrow} V(\tau), \quad \tau \in \mathbb{R}_+. \quad (1.2)$$

The classical Lamperti’s theorem [26] says that in the case of (1.2), the normalizing constants  $B_n$  necessarily grow as  $n^H$  (modulus a slowly varying factor) and the limit random process in (1.2) is  $H$ -sssi. The limit process  $\{V(\tau)\}$  in (1.2) characterizes large-scale and dependence properties of  $\{Y(t)\}$ , leading to the important concept of *distributional short/long memory* (Cox [10]). There exists a large probabilistic literature devoted to studying the partial sums limits

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of various classes of strongly and weakly dependent processes and random fields. See, e.g., the monographs [4], [14], [18] and the references therein. In particular, several works ([11], [12], [30], [41], [13]) discussed the partial sums limits of (stationary) random fields indexed by  $t \in \mathbb{Z}^d$ :

$$B_n^{-1} \sum_{t \in K_{[nx]}} Y(t) \xrightarrow{\text{fdd}} V(x), \quad x = (x_1, \dots, x_d) \in \mathbb{R}_+^d, \quad (1.3)$$

where  $K_{[nx]} := \{t = (t_1, \dots, t_d) \in \mathbb{Z}^d : 1 \leq t_i \leq nx_i\}$  is a sequence of rectangles whose all sides increase as  $O(n)$ . Related results for Gaussian or linear (shot-noise) and their subordinated random fields, with a particular emphasis on large-time behavior of statistical solutions of partial differential equations, were obtained in [1], [2], [30], [32]. See also the recent paper Anh et al. [3] and the numerous references therein. Most of the above mentioned studies deal with ‘nearly isotropic’ models of random fields characterized by a single memory parameter  $H$  and a limiting SSRF  $\{V(x)\}$  in (1.3).

The present paper attempts a systematic study of *anisotropic distributional long memory*, by exhibiting a natural class of models whose partial sums tend to OSRFs. Related notion of *anisotropic long memory in spectral domain* and its implications is discussed in Lavancier et al. [29]. The present study is limited to the case  $d = 2$  and random fields with the horizontal anisotropy axis and a diagonal matrix  $E$ . Note that for  $d = 2$  and  $E = \text{diag}(1, \gamma)$ ,  $0 < \gamma \neq 1$ , relation (1.1) writes as  $\{V(\lambda x, \lambda^\gamma y)\} \xrightarrow{\text{fdd}} \{\lambda^H V(x, y)\}$ ,  $(x, y) \in \mathbb{R}^2$ , or

$$\{\lambda V(x, y)\} \xrightarrow{\text{fdd}} \{V(\lambda^{1/H_1} x, \lambda^{1/H_2} y)\}, \quad \forall \lambda > 0, \quad (1.4)$$

where  $H_1 := H, H_2 := H/\gamma \neq H_1$ . The OSRFs (1.4) discussed in our paper are obtained by taking the partial sums limits

$$B_n^{-1} \sum_{(t,s) \in K_{[nx, n^{H_1/H_2} y]}} Y(t, s) \xrightarrow{\text{fdd}} V(x, y), \quad (x, y) \in \mathbb{R}_+^2 \quad (1.5)$$

on *incommensurate* rectangles  $K_{[nx, n^{H_1/H_2} y]} := \{(t, s) \in \mathbb{Z}^2 : 1 \leq t \leq nx, 1 \leq s \leq n^{H_1/H_2} y\}$  with sides growing at different rates  $O(n)$  and  $O(n^{H_1/H_2})$ . The convergence in (1.5) is established for a natural class of aggregated random-coefficient autoregressive random fields, see (1.6)–(1.9) below, with finite and infinite variance. The idea of contemporaneous aggregation originates to Granger [20], who observed that aggregation of random-coefficient AR(1) equations with random beta-distributed coefficient can lead to a Gaussian process with long memory and slowly decaying covariance function. Since then, aggregation became of the most important methods for modeling and studying long memory processes (Beran [4]). For linear and heteroscedastic autoregressive time series models with one-dimensional time it was developed in [19], [40], [21], [44], [45], [8], [17], [37], [38], [36], and for some random field models in [27], [28], [29], [43]. Aggregation is also important for understanding and modeling of spatial long memory processes by relating them to short-memory random-coefficient autoregressive models in a natural way. On the other hand, the proof of (1.5) for aggregated autoregressive fields is not easy and requires a complete control of the corresponding autoregressive model near the ‘unit root boundary’.

Consider a nearest-neighbor autoregressive random field  $\{X(t, s)\}$  on  $\mathbb{Z}^2$  satisfying the difference equation

$$X(t, s) = \sum_{|u|=|v|=1} a(u, v) X(t+u, s+v) + \varepsilon(t, s), \quad (t, s) \in \mathbb{Z}^2, \quad (1.6)$$

where  $\{\varepsilon(t, s), (t, s) \in \mathbb{Z}^2\}$  are i.i.d. r.v.’s whose generic distribution  $\varepsilon$  belongs to the domain of (normal) attraction of  $\alpha$ -stable law,  $0 < \alpha \leq 2$ , and  $a(t, s) \geq 0, |t| = |s| = 1$  are *random* coefficients, independent of  $\{\varepsilon(t, s)\}$  and satisfying the following condition for the existence of a stationary solution of (1.6):

$$A := \sum_{|t|=|s|=1} a(t, s) < 1, \quad \text{a.s.} \quad (1.7)$$

The stationary solution of (1.6) is given by the convergent series

$$X(t, s) = \sum_{(u,v) \in \mathbb{Z}^2} g(t-u, s-v, a) \varepsilon(u, v), \quad (t, s) \in \mathbb{Z}^2, \quad (1.8)$$

where  $g(t, s, a)$ ,  $(t, s) \in \mathbb{Z}^2$ ,  $a = (a(t, s), |t| = |s| = 1)$ , is the (random) lattice Green function solving the equation  $g(t, s, a) - \sum_{|u|=|v|=1} a(u, v)g(t+u, s+v, a) = \delta(t, s)$ , where  $\delta(t, s)$  is the delta function (see sec. 2 for precise statement). Let  $\{X_i(t, s)\}$ ,  $i = 1, 2, \dots$  be independent copies of (1.8). The aggregated field  $\{\mathfrak{X}(t, s)\}$  is defined as the limit in distribution:

$$N^{-1/\alpha} \sum_{i=1}^N X_i(t, s) \xrightarrow{\text{fdd}} \mathfrak{X}(t, s), \quad (t, s) \in \mathbb{Z}^2. \quad (1.9)$$

Let  $\Phi$  denote the distribution of the random vector  $a = (a(t, s), |t| = |s| = 1)$  taking values in  $\mathbf{A} := \{a_{t,s} \in [0, 1], \sum_{|t|=|s|=1} a_{t,s} < 1\} \subset \mathbb{R}^4$  and called below the *mixing distribution*. Under mild additional conditions, the limit in (1.9) exists and is written as

$$\mathfrak{X}(t, s) = \sum_{(u,v) \in \mathbb{Z}^2} \int_{\mathbf{A}} g(t-u, s-v, a) M_{u,v}(da), \quad (t, s) \in \mathbb{Z}^2. \quad (1.10)$$

In (1.10),  $\{M_{u,v}(da), (u, v) \in \mathbb{Z}^2\}$  are i.i.d. copies of an  $\alpha$ -stable random measure  $M$  on  $\mathbf{A}$  with control measure  $\Phi$ , see (3.16). The random field  $\{\mathfrak{X}(t, s)\}$  in (1.10) is  $\alpha$ -stable and a particular case of mixed stable moving-average fields introduced in [42]. In the case  $\alpha = 2$ , or a Gaussian limit in (1.10), the covariance function and the spectral density of this random field are given by

$$r(t, s) = \sigma^2 \sum_{(u,v) \in \mathbb{Z}^2} E g(u, v, a) g(t+u, s+v, a), \quad (t, s) \in \mathbb{Z}^2, \quad (1.11)$$

and

$$f(x, y) = (\sigma^2 / (2\pi)^2) E |\hat{g}(x, y, a)|^2, \quad (x, y) \in [-\pi, \pi]^2, \quad (1.12)$$

respectively, where  $\hat{g}(x, y, a) = (1 - \sum_{|t|=|s|=1} a(t, s) e^{i(xt+ys)})^{-1}$  is the Fourier transform of  $g(t, s, a)$  and  $\sigma^2 := E\varepsilon^2$ .

It is not surprising that large-scale and long memory properties of the aggregated field  $\{\mathfrak{X}(t, s)\}$  strongly depend on the behavior of  $\Phi$  near the ‘unit root’  $A = 1$ . We assume in Sections 4, 5 and 6 that the ‘angular coefficients’  $0 \leq p(t, s) := a(t, s)/A$ ,  $\sum_{|t|=|s|=1} p(t, s) = 1$  are nonrandom and the ‘radial coefficient’  $A \in [0, 1)$  is random and has a regularly varying probability density  $\phi$  at  $a = 1$ :

$$\phi(a) \sim \phi_1 (1-a)^\beta, \quad a \nearrow 1, \quad \exists \phi_1 > 0, \quad 0 < \beta < \alpha - 1, \quad 1 < \alpha \leq 2. \quad (1.13)$$

(The case  $0 < \alpha < 1$  apparently cannot lead to long-range dependence; see [37], [38]). The three models of interest are given by the following equations:

$$X(t, s) = \frac{A}{2} (X(t-1, s) + X(t, s-1)) + \varepsilon(t, s), \quad (1.14)$$

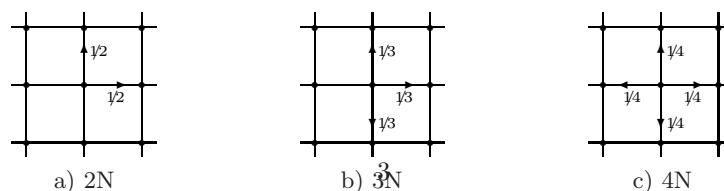
$$X(t, s) = \frac{A}{3} (X(t-1, s) + X(t, s+1) + X(t, s-1)) + \varepsilon(t, s), \quad (1.15)$$

$$X(t, s) = \frac{A}{4} (X(t-1, s) + X(t+1, s) + X(t, s+1) + X(t, s-1)) + \varepsilon(t, s). \quad (1.16)$$

In the sequel, we refer to (1.14), (1.15) and (1.16) as 2N, 3N and 4N models, N standing for ‘Neighbor’. Stationary solution of the above equations in all three cases is given by (1.8), the Green function being written as

$$g(t, s, a) = \sum_{k=0}^{\infty} A^k p_k(t, s), \quad (t, s) \in \mathbb{Z}^2, \quad a \in \mathbf{A}, \quad (1.17)$$

where  $p_k(t, s) = P(W_k = (t, s) | W_0 = (0, 0))$  is the  $k$ -step probability of the nearest-neighbor random walk  $\{W_k, k = 0, 1, \dots\}$  on the lattice  $\mathbb{Z}^2$  with one-step transition probabilities  $p(t, s)$  shown in Figure 1 a) - c).



Relation (1.12) implies (see also Remark 3.2 below) that for all three models (2N, 3N, and 4N),  $\alpha = 2$ , and a mixing density as in (1.13), the aggregated spectral density  $f(x, y)$  in (1.12) is unbounded for all  $0 < \beta < 1$ , meaning that the corresponding Gaussian random field in (1.10) has long memory. [29] obtained the asymptotics of  $f(x, y)$  as  $(x, y) \rightarrow (0, 0)$  in an arbitrary way and showed that the 2N and 3N models satisfy spectral anisotropic long memory property (a spectral analog of the anisotropic distributional long memory property of Definition 2.2), in contrast to the 4N model having isotropic long memory spectrum ([28], [29]). The above mentioned works use the spectral approach which is applicable in the case  $\alpha = 2$  only. Asymptotics of spectral density and covariance functions for some long-range dependent random fields was also studied in [31].

The present paper is based on a direct study of the asymptotics of the lattice Green function in (1.17) for models 2N, 3N, and 4N, using classical probabilistic tools (the Moivre-Laplace theorem and the Hoeffding inequality for tails of binomial distribution, see [15], [16], [23]). In particular, Lemmas 4.1, 5.1 and 6.1 obtain the following point-wise convergences: as  $\lambda \rightarrow \infty$ ,

$$\sqrt{\lambda}g_2\left(\frac{[\lambda t] + [\sqrt{\lambda}s]}{2}, \frac{[\lambda t] - [\sqrt{\lambda}s]}{2}, 1 - \frac{z}{\lambda}\right)\mathbf{1}([\lambda t] \equiv [\sqrt{\lambda}s] \pmod{2}) \rightarrow h_2(t, s, z), \quad t > 0, s \in \mathbb{R}, z > 0, \quad (1.18)$$

$$\sqrt{\lambda}g_3([\lambda t], [\sqrt{\lambda}s], 1 - \frac{z}{\lambda}) \rightarrow h_3(t, s, z), \quad t > 0, s \in \mathbb{R}, z > 0, \quad (1.19)$$

$$g_4([\lambda t], [\lambda s], 1 - \frac{z}{\lambda^2}) \rightarrow h_4(t, s, z), \quad (t, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}, z > 0, \quad (1.20)$$

respectively, together with dominating bounds of the left-hand sides of (1.18)-(1.20), see (4.8), (5.5), and (6.5), below. Here and in the rest of the paper,  $g_2$ ,  $g_3$  and  $g_4$  denote the Green functions of the 2N, 3N and 4N models, respectively, and the limit functions  $h_2$ ,  $h_3$  and  $h_4$  in (1.18)-(1.20) are given by

$$\begin{aligned} h_2(t, s, z) &:= \sqrt{\frac{2}{\pi t}} e^{-zt - \frac{s^2}{2t}}, & h_3(t, s, z) &:= \frac{3}{2\sqrt{\pi t}} e^{-3zt - \frac{s^2}{4t}}, \\ h_4(t, s, z) &:= \frac{2}{\pi} K_0(2\sqrt{z(t^2 + s^2)}), \end{aligned} \quad (1.21)$$

where  $K_0$  is the modified Bessel function of second kind. Note that  $h_2$  and  $h_3$  in (1.21) are the Green function of one-dimensional heat equation (modulus constant coefficients), while  $h_4$  is the Green function of the Helmholtz equation in  $\mathbb{R}^2$ . The obvious similarity between kernels  $h_2$  and  $h_3$  suggest that large-scale properties of the 2N and 3N models should be similar, modulus a rotation of the plane by angle  $\pi/4$ . Kernels  $h_2$ ,  $h_3$  and  $h_4$  appear in the stochastic integral representation of scaling limits of models (1.14)-(1.16).

Lemmas 4.1, 5.1 and 6.1 play a crucial role in our study of long memory properties of the aggregated field  $\{\mathfrak{X}(t, s)\}$  (1.10) and present the main technical difficulty of this paper. These lemmas may have independent interest for studying the behavior of the autoregressive fields (1.14), (1.15) and (1.16) with deterministic coefficient  $A$  in the vicinity of  $A = 1$ . Particularly, we expect that these lemmas can be used for testing of stationarity and coefficient estimation near the unit root  $A = 1$  in spatial autoregressive models (1.14), (1.15) and (1.16), c.f. [5], [7], [34], [35].

Let us summarize the remaining contents of the paper. Sec. 2 introduces the notions of anisotropic/isotropic distributional long memory, in terms of scaling behavior of partial sums limits (1.5)/(1.3). An important feature of Definitions 2.2 and 2.3 is the requirement of dependence of increments of the limit random field in arbitrary direction. This requirement is analogous to the dependence of increments requirement in the definition of distributional long memory for processes indexed by  $t \in \mathbb{Z}$  in [10], and helps to separate between isotropic and anisotropic scaling behaviors. See also Proposition 5.1.

Sec. 3 discusses the existence of stationary solution in  $L^p$  ( $0 < p \leq 2$ ) of the nearest-neighbor random-coefficient equation (1.6), and the aggregated limit in (1.9) as a mixed  $\alpha$ -stable moving average field of (1.10). Sec. 4, 5 and 6 are devoted to the study of scaling limits of the aggregated 2N, 3N and 4N models, respectively. The convergence in (1.5) with  $B_n = n^{H_1}$ ,  $H_1 := \frac{\frac{1}{2} + \alpha - \beta}{\alpha}$ ,  $H_2 := 2H_1$  and the anisotropic long memory property are established in Theorem 5.1 for the aggregated 3N model  $\{\mathfrak{X}(t, s) \equiv \mathfrak{X}_3(t, s)\}$  of (1.10). The limit random field  $\{V_3(x, y)\}$  is an  $\alpha$ -stable OSRF and satisfies (1.1). It is represented in (5.2) as a stochastic integral with respect to an  $\alpha$ -stable random measure with integrand involving the kernel  $h_3$  in (1.21). For the same random field  $\{\mathfrak{X}_3(t, s)\}$ , Theorem 5.2 obtains a ‘commensurate’

scaling limit of (1.3) towards a different random field  $\{V_{3*}(x, y)\}$  in (5.16), which is self-similar with  $H_* := \frac{1+\alpha-\beta}{\alpha}$  and has independent increments in the vertical direction (see Definition 2.1). Similar results as for the 3N model are obtained for the aggregated 2N model  $\{\mathfrak{X}_2(t, s)\}$  (Theorems 4.1, 4.2 and Proposition 4.2). In the finite variance case  $\alpha = 2$ , Proposition 5.2 obtains the asymptotic decay of the covariance  $r_3(t, s) = E[\mathfrak{X}_3(0, 0)\mathfrak{X}_3(t, s)]$  as  $t \rightarrow \infty$  and  $s = O(\sqrt{t})$  increase ‘parabolically’, complementing the result in [29] on anisotropic asymptotics of the spectral density.

Sec. 6 discusses the lattice isotropic aggregated 4N model  $\{\mathfrak{X}_4(t, s)\}$ . We show that this field satisfies the isotropic distributional long memory property of Definition 2.3 and its scaling limit  $\{V_4(x, y)\}$  is an  $\alpha$ -stable SSRF with exponent  $H = \frac{2(\alpha-\beta)}{\alpha}$ , see Theorem 6.1 and Proposition 6.1. The isotropic covariance long memory property for  $\{\mathfrak{X}_4(t, s)\}$  and  $\alpha = 2$  is proved in Proposition 6.2. In the Gaussian case  $\alpha = 2$ , Theorem 6.1 and Proposition 6.2 agree with [28]. Sec. 7 (Appendix) contains the proofs of the technical Lemmas 4.1, 5.1 and 6.1.

*Notation.* In what follows,  $C, C(K), \dots$  denote generic constants, possibly depending on the variables in brackets, which may be different at different locations. We write  $\xrightarrow{d}, \stackrel{d}{=}, \xrightarrow{\text{fdd}}, \stackrel{\text{fdd}}{=}$  for the weak convergence and equality of distributions and finite-dimensional distributions, respectively. fdd-lim stands for the limit in the sense of weak convergence of finite-dimensional distributions. For  $\lambda > 0$  and a  $d \times d$  matrix  $E$ ,  $\lambda^E := e^{E \log \lambda}$ , where  $e^A = \sum_{k=0}^{\infty} A^k / k!$  is the matrix exponential.  $\mathbb{Z}_+^d := \{(t_1, \dots, t_d) \in \mathbb{Z}^d : t_i \geq 0, i = 1, \dots, d\}$ ,  $\mathbb{R}_+^d := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \geq 0, i = 1, \dots, d\}$ ,  $\mathbb{Z}_+ := \mathbb{Z}_+^1$ ,  $\mathbb{R}_+ := \mathbb{R}_+^1$ .  $E = \text{diag}(\gamma_1, \dots, \gamma_d)$  denotes the diagonal  $d \times d$  matrix with entries  $\gamma_1, \dots, \gamma_d$  on the diagonal. For integers  $t, s$ ,  $t \stackrel{\text{mod } 2}{=} s$  and  $t \stackrel{\text{mod } 2}{\neq} s$  means that  $t + s$  is even and odd, respectively. All equalities and inequalities between random variables are assumed to hold almost surely.

## 2 Isotropic and anisotropic long memory of random fields in $\mathbb{Z}^2$

Let  $\ell = \{(x, y) \in \mathbb{R}^2 : ax + by = c\}$  be a line in  $\mathbb{R}^2$ . A line  $\ell' = \{(x, y) \in \mathbb{R}^2 : a'x + b'y = c'\}$  is said *perpendicular to*  $\ell$  (denoted  $\ell' \perp \ell$ ) if  $aa' + bb' = 0$ . Write  $(u, v) \prec (x, y)$  (respectively,  $(u, v) \preceq (x, y)$ ),  $(u, v), (x, y) \in \mathbb{R}_+^2$ , if  $u < x$  and  $v < y$  (respectively,  $u \leq x$  and  $v \leq y$ ) hold. A *rectangle* is a set  $K_{(u,v);(x,y)} := \{(s, t) \in \mathbb{R}_+^2 : (u, v) \prec (s, t) \preceq (x, y)\}$ ;  $K_{x,y} := K_{(0,0);(x,y)}$ . We say that two rectangles  $K = K_{(u,v);(x,y)}$  and  $K' = K_{(u',v');(x',y')}$  are *separated by line*  $\ell$  if they lie on different sides of  $\ell$ , in which case  $K$  and  $K'$  are necessarily disjoint:  $K \cap K' = \emptyset$  (see Fig. 2 below).

Let  $\{V(x, y)\} = \{V(x, y), (x, y) \in \mathbb{R}_+^2\}$  be a random field and  $K = K_{(u,v);(x,y)} \subset \mathbb{R}_+^2$  be a rectangle. By *increment of*  $\{V(x, y)\}$  *on rectangle*  $K$  we mean the difference

$$V(K) := V(x, y) - V(u, y) - V(x, v) + V(u, v).$$

**Definition 2.1** Let  $\{V(x, y), (x, y) \in \mathbb{R}_+^2\}$  be a random field with  $V(x, 0) = V(0, y) \equiv 0$ ,  $x, y \geq 0$ , and  $\ell \subset \mathbb{R}^2$ ,  $(0, 0) \in \ell$  be a given line passing through the origin. We say that  $\{V(x, y)\}$  has independent increments in direction  $\ell$  if for any orthogonal line  $\ell' \perp \ell$  and any two rectangles  $K, K' \subset \mathbb{R}_+^2$  separated by  $\ell'$ , increments  $V(K)$  and  $V(K')$  are independent. Else, we say that  $\{V(x, y)\}$  has dependent increments in direction  $\ell$ .

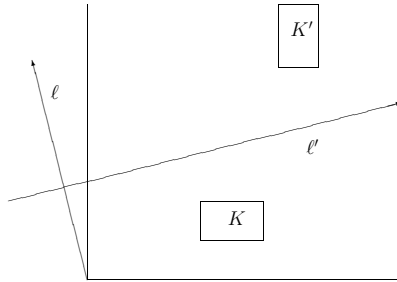


FIGURE 2

**Definition 2.2** We say that a stationary random field  $\{Y(t, s), (t, s) \in \mathbb{Z}^2\}$  has anisotropic distributional long memory with parameters  $H_1, H_2 > 0, H_1 \neq H_2$  if

$$n^{-H_1} \sum_{(t,s) \in K_{[nx, n^{H_1/H_2}y]}} Y(t, s) \xrightarrow{\text{fdd}} V(x, y), \quad (x, y) \in \mathbb{R}_+^2, \quad (2.1)$$

where  $\{V(x, y)\}$  is a random field having dependent increments in arbitrary direction.

**Definition 2.3** We say that a stationary random field  $\{Y(t, s), (t, s) \in \mathbb{Z}^2\}$  has isotropic distributional long memory with parameter  $H > 0$  if

$$n^{-H} \sum_{(t,s) \in K_{[nx, ny]}} Y(t, s) \xrightarrow{\text{fdd}} V(x, y), \quad (x, y) \in \mathbb{R}_+^2, \quad (2.2)$$

where  $\{V(x, y)\}$  is a random field having dependent increments in arbitrary direction.

**Proposition 2.1** (i) Let  $\{Y(t, s)\}$  have anisotropic distributional long memory with parameters  $H_1 \neq H_2$ . Then the limit random field  $\{V(x, y)\}$  in (2.1) satisfies the self-similarity property (1.4). In particular,  $\{V(x, y)\}$  is OSRF corresponding to  $H := H_1, E := \text{diag}(1, H_1/H_2)$ .

(ii) Let  $\{Y(t, s)\}$  have isotropic distributional long memory with parameter  $H$ . Then the limit random field  $\{V(x, y)\}$  in (2.2) satisfies the self-similarity property (1.4) with  $H_1 = H_2 := H$ , i.e.,  $\{V(x, y)\}$  is SSRF with parameter  $H$ .

*Proof.* Fix  $\lambda > 0$  and let  $m := [n\lambda^{1/H_1}]$ . We have

$$\begin{aligned} V(\lambda^{1/H_1}x, \lambda^{1/H_2}y) &= \text{fdd-lim} \frac{1}{n^{H_1}} \sum_{0 < t \leq x\lambda^{1/H_1}n, 0 < s \leq y\lambda^{1/H_2}n^{H_1/H_2}} Y(t, s) \\ &= \text{fdd-lim} \frac{\lambda}{m^{H_1}} \sum_{0 < t \leq xm, 0 < s \leq ym^{H_1/H_2}} Y(t, s) \\ &\stackrel{\text{fdd}}{=} \lambda V(x, y). \end{aligned}$$

Proposition 2.1 is proved. □

### 3 The existence of the limit aggregated autoregressive random field

We first discuss the solvability of the nearest-neighbor random-coefficient autoregressive equation (1.6) and the convergence of the series (1.8). The Green function of (1.6) is written as

$$g(t, s, a) = \sum_{k=0}^{\infty} a^{\star k}(t, s), \quad (3.1)$$

where  $a^{\star k}(t, s)$  is the  $k$ -fold convolution of the function  $a(t, s), (t, s) \in \mathbb{Z}^2, a(t, s) := 0$  ( $|t| + |s| \neq 1$ ) defined recursively by

$$a^{\star 0}(t, s) = \delta(t, s) := \begin{cases} 1, & (t, s) = (0, 0), \\ 0, & (t, s) \neq (0, 0) \end{cases}, \quad a^{\star k}(t, s) = \sum_{(u,v) \in \mathbb{Z}^2} a^{\star (k-1)}(u, v) a(t-u, s-v), \quad k \geq 1.$$

Note that (3.1) can be rewritten as (1.17), where  $p_k(t, s) = P(W_k = (t, s) | W_0 = (0, 0))$  is the  $k$ -step probability of the nearest-neighbor random walk  $\{W_k, k = 0, 1, \dots\}$  on  $\mathbb{Z}^2$  with one-step transition probabilities

$$p(t, s) \equiv p(t, s, a) = P(W_1 = (t, s) | W_0 = (0, 0)) := \frac{a(t, s)}{A}, \quad (t, s) \in \mathbb{Z}^2, \quad p(t, s) := 0 \quad (|t| + |s| \neq 1). \quad (3.2)$$

Generally, the  $p_k(t, s)$ 's depend also on  $a = (a_{t,s}, |t| = |s| = 1) \in \mathbf{A}$  but this dependence is suppressed for brevity. Write  $\varepsilon$  for generic  $\varepsilon(t, s), (t, s) \in \mathbb{Z}^2$ . Let

$$q_1 := p(0, 1) + p(0, -1) = 1 - p(1, 0) - p(-1, 0) =: 1 - q_2, \quad q := \min(q_1, q_2). \quad (3.3)$$

Note  $q_i \in [0, 1]$  and  $q_1 = 0$  (respectively,  $q_2 = 0$ ) means that the random walk  $\{W_k\}$  is concentrated on the horizontal (respectively, vertical) axis of the lattice  $\mathbb{Z}^2$ .

**Proposition 3.1** (i) Assume there exists  $0 < p \leq 2$  such that

$$\mathbb{E}|\varepsilon|^p < \infty \quad \text{and} \quad \mathbb{E}\varepsilon = 0 \quad \text{for} \quad 1 \leq p \leq 2, \quad (3.4)$$

and condition (1.7). Then there exists a stationary solution of random-coefficient equation (1.6) given by (1.8), where the series converges conditionally a.s. and in  $L^p$  for every  $a \in \mathbf{A}$ .

(ii) In addition to (3.4), assume that  $q > 0$  a.s. and

$$\begin{cases} \mathbb{E}\left[\frac{1}{q^{2(p-1)}(1-A)}\right] < \infty, & \text{if } 1 < p \leq 2, \\ \mathbb{E}\left[\frac{1}{(1-A)^{3-2p}}\right] < \infty, & \text{if } 0 < p \leq 1. \end{cases} \quad (3.5)$$

Then the series in (1.8) converges unconditionally in  $L^p$ .

*Proof.* (i) Let us prove the convergence of (1.8). We shall use the following inequality. Let  $0 < p \leq 2$ , and let  $\xi_1, \xi_2, \dots$  be random variables with  $\mathbb{E}|\xi_i|^p < \infty$ . For  $1 \leq p \leq 2$ , assume in addition that the  $\xi_i$ 's are independent and have zero mean  $\mathbb{E}\xi_i = 0$ . Then

$$\mathbb{E}\left|\sum_i \xi_i\right|^p \leq 2 \sum_i \mathbb{E}|\xi_i|^p. \quad (3.6)$$

Accordingly,

$$\mathbb{E}\left[|X(t, s)|^p | a\right] \leq 2\mathbb{E}|\varepsilon|^p \sum_{(u,v) \in \mathbb{Z}^2} |g(u, v, a)|^p. \quad (3.7)$$

By (1.17),

$$0 \leq g(t, s, a) \leq \sum_{k=|t|+|s|}^{\infty} A^k p_k(t, s) \leq \frac{A^{|t|+|s|}}{1-A} \quad (3.8)$$

From above we obtain

$$\mathbb{E}\left[|X(t, s)|^p | a\right] \leq C \sum_{(u,v) \in \mathbb{Z}^2} A^{p(|u|+|v|)} \leq C \sum_{k=0}^{\infty} A^{pk} (4k+1) < \infty, \quad (3.9)$$

proving the conditional convergence in  $L^p$  of the series in (1.8).

Let prove part (ii). According to the bound in (3.7), it suffices to prove that

$$\mathbb{E} \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)|^p < \infty. \quad (3.10)$$

Let  $\hat{a}(x, y) := \sum_{|t|+|s|=1} e^{-i(tx+sy)} a(t, s)$ ,  $(x, y) \in \Pi^2$ ,  $\Pi := [-\pi, \pi]$ . Then  $a(t, s) = (2\pi)^{-2} \int_{\Pi^2} e^{i(tx+sy)} \hat{a}(x, y) dx dy$  and

$$g(t, s, a) = \frac{1}{(2\pi)^2} \int_{\Pi^2} e^{i(tx+sy)} \frac{dx dy}{1 - \hat{a}(x, y)} = \frac{1}{(2\pi)^2} \int_{\Pi^2} e^{i(tx+sy)} \frac{dx dy}{1 - A\hat{p}(x, y)},$$

where  $\hat{p}(x, y) := \hat{a}(x, y)/A = \sum_{|t|+|s|=1} e^{-i(tx+sy)} p(t, s)$  satisfies  $|\hat{p}(x, y)| \leq \sum_{|t|+|s|=1} p(t, s) = 1$ . From Parseval's identity,

$$\sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)|^2 = C \int_{\Pi^2} \frac{dx dy}{|1 - A\hat{p}(x, y)|^2}. \quad (3.11)$$



We shall need the inequality

$$|1 - A\hat{p}(x, y)| \geq (1/24)q[(1 - A) + x^2 + y^2], \quad (x, y) \in \Pi^2, \quad (3.12)$$

which is proved below. We have

$$\begin{aligned} 1 - A\hat{p}(x, y) &= (1 - A) + A \sum_{|t|+|s|=1} p(t, s)(1 - e^{i(tx+sy)}) \\ &= (1 - A) + A[q_2(1 - \cos(x)) + q_1(1 - \cos(y))] - iA \sum_{|t|=|s|=1} p(t, s) \sin(tx + sy) \end{aligned}$$

and therefore

$$|1 - A\hat{p}(x, y)| \geq (1 - A) + Aq[(1 - \cos(x)) + (1 - \cos(y))],$$

proving (3.12) (we used the inequalities  $1 - \cos(x) \geq x^2/8$  and  $x^2 \leq 10$ ,  $|x| \leq \pi$ ). From (3.11) and (3.12) we obtain

$$\sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)|^2 \leq \frac{C}{q^2} \int_{\Pi^2} \frac{dx dy}{((1 - A) + x^2 + y^2)^2} \leq \frac{C}{q^2} \int_0^\infty \frac{r dr}{((1 - A) + r^2)^2} = \frac{C}{q^2(1 - A)}. \quad (3.13)$$

On the other hand, (1.17) immediately gives  $\sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)| = \sum_{k=0}^\infty A^k \sum_{(t,s) \in \mathbb{Z}^2} p_k(t, s) = \sum_{k=0}^\infty A^k = \frac{1}{1-A}$ . Therefore for any  $1 < p < 2$ , by Hölder's inequality,

$$\begin{aligned} \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)|^p &\leq \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)|^{2(p-1)} |g(t, s, a)|^{2-p} \mathbf{1}(|g(t, s, a)| > 1) + \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)| \mathbf{1}(|g(t, s, a)| \leq 1) \\ &\leq \left( \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)|^2 \right)^{p-1} \left( \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)| \mathbf{1}(|g(t, s, a)| > 1) \right)^{2-p} + \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)| \\ &\leq C \left( \frac{1}{q^2(1 - A)} \right)^{p-1} \left( \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)| \right)^{2-p} + \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)| \\ &\leq \frac{C}{q^{2(p-1)}(1 - A)} + \frac{C}{1 - A} \leq \frac{C}{q^{2(p-1)}(1 - A)}, \end{aligned}$$

proving (3.10) and the unconditional convergence of (1.8) under the first condition in (3.5).

Next, consider the case  $0 < p \leq 1$ . Using (1.17) and Hölder's inequality, we obtain

$$\begin{aligned} \sum_{(t,s) \in \mathbb{Z}^2} |g(t, s, a)|^p &\leq \sum_{k=0}^\infty A^{kp} \sum_{|t|+|s| \leq k} p_k^p(t, s) \\ &\leq \sum_{k=0}^\infty A^{kp} \left\{ \sum_{|t|+|s| \leq k} p_k(t, s) \right\}^p \left\{ \sum_{|t|+|s| \leq k} 1 \right\}^{1-p} \\ &\leq C \sum_{k=0}^\infty A^{kp} k^{2(1-p)} \leq \frac{C}{(1 - A^p)^{3-2p}} \leq \frac{C}{(1 - A)^{3-2p}}. \end{aligned}$$

This completes the proof of part (ii) and the proposition. □

**Definition 3.1** Write  $\varepsilon \in D(\alpha)$ ,  $0 < \alpha \leq 2$ , if

- (i)  $\alpha = 2$  and  $E\varepsilon = 0$ ,  $\sigma^2 := E\varepsilon^2 < \infty$ , or
- (ii)  $0 < \alpha < 2$  and there exist some constants  $c_1, c_2 \geq 0$ ,  $c_1 + c_2 > 0$ , such that

$$\lim_{x \rightarrow \infty} x^\alpha P(\varepsilon > x) = c_1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} |x|^\alpha P(\varepsilon \leq x) = c_2;$$

moreover,  $E\varepsilon = 0$  whenever  $1 < \alpha < 2$ , while, for  $\alpha = 1$ , we assume that the distribution of  $\varepsilon$  is symmetric.



**Remark 3.1** Condition  $\varepsilon \in D(\alpha)$  implies that the r.v.  $\varepsilon$  belongs to the domain of normal attraction of an  $\alpha$ -stable law; in other words,

$$N^{-1/\alpha} \sum_{i=1}^N \varepsilon_i \xrightarrow{d} Z, \quad (3.14)$$

where  $Z$  is an  $\alpha$ -stable r.v., see ([16], pp.574-581). The symmetry of  $\varepsilon$  in the case  $\alpha = 1$  is not necessary for (3.14) and the subsequent discussion; however, it is imposed to avoid some technical and notational complications. The characteristic function of the r.v.  $Z$  in (3.14) is given by

$$\mathbb{E} e^{i\theta Z} = e^{-|\theta|^\alpha \omega(\theta)}, \quad \theta \in \mathbb{R},$$

where

$$\omega(\theta) := \begin{cases} \frac{\Gamma(2-\alpha)}{1-\alpha} \left( (c_1 + c_2) \cos\left(\frac{\pi\alpha}{2}\right) - i(c_1 - c_2) \text{sign}(\theta) \sin\left(\frac{\pi\alpha}{2}\right) \right), & \alpha \neq 1, 2, \\ (c_1 + c_2) \frac{\pi}{2}, & \alpha = 1, \\ \frac{\sigma^2}{2}, & \alpha = 2. \end{cases} \quad (3.15)$$

Introduce independently scattered  $\alpha$ -stable random measure  $M$  on  $\mathbb{Z}^2 \times \mathbf{A}$  with characteristic functional

$$\mathbb{E} \exp \left\{ i \sum_{(t,s) \in \mathbb{Z}^2} \theta_{t,s} M_{t,s}(A_s) \right\} = \exp \left\{ - \sum_{(t,s) \in \mathbb{Z}^2} |\theta_{t,s}|^\alpha \omega(\theta_{t,s}) \Phi(A_{t,s}) \right\}, \quad (3.16)$$

where  $\theta_{t,s} \in \mathbb{R}$  and  $A_{t,s} \subset \mathbf{A}$  are arbitrary Borel sets.

**Proposition 3.2** *Let  $\varepsilon \in D(\alpha)$ ,  $0 < \alpha \leq 2$ . Assume that the mixing distribution satisfies condition (3.5) of Proposition 3.1 (ii) with some  $0 < p \leq 2$  and such that*

$$\begin{cases} p > \alpha, & \text{if } 1 < \alpha < 2, \\ p < \alpha, & \text{if } 0 < \alpha < 1, \\ p = 2, & \text{if } \alpha = 2. \end{cases} \quad (3.17)$$

*In the case  $\alpha = 1$  we assume that*

$$\mathbb{E} \frac{1}{(1-A)^p} < \infty \quad \text{for some } p > 1. \quad (3.18)$$

*Then the limit aggregated random field in (1.9) exists and has the stochastic integral representation of (1.10).*

*Proof.* Let  $T \subset \mathbb{Z}^2$  be a finite set,  $\theta_{t,s} \in \mathbb{R}$ ,  $(t,s) \in T$ . It suffices to prove that  $S_N \xrightarrow{d} S$ , where  $S := \sum_{(t,s) \in T} \theta_{t,s} \mathfrak{X}(t,s)$  is a  $\alpha$ -stable r.v. with characteristic function

$$\mathbb{E} e^{i w S} = \exp \left\{ - |w|^\alpha \sum_{(u,v) \in \mathbb{Z}^2} \mathbb{E} \left[ |G(u,v,a)|^\alpha \omega(w G(u,v,a)) \right] \right\}, \quad G(u,v,a) := \sum_{(t,s) \in T} \theta_{t,s} g(t-u, s-v, a)$$

and  $S_N = N^{-1/\alpha} \sum_{i=1}^N U_i$  is a sum of i.i.d. r.v.'s with common distribution

$$U := \sum_{(t,s) \in T} \theta_{t,s} X(t,s) = \sum_{(u,v) \in \mathbb{Z}^2} G(u,v,a) \varepsilon(u,v).$$

It suffices to prove that r.v.  $U$  belongs to the domain of attraction of r.v.  $S$  (in the sense of (3.14)); in other words, that

$$\mathbb{E} U^2 = \mathbb{E} S^2 < \infty \quad \text{for } \alpha = 2, \quad (3.19)$$

and, for  $0 < \alpha < 2$ ,

$$\begin{aligned} \lim_{x \rightarrow \infty} x^\alpha \mathbb{P}(U > x) &= \sum_{(u,v) \in \mathbb{Z}^2} \mathbb{E} \left[ |G(u,v,a)|^\alpha \{ c_1 \mathbf{1}(G(u,v,a) > 0) + c_2 \mathbf{1}(G(u,v,a) < 0) \} \right], \\ \lim_{x \rightarrow -\infty} |x|^\alpha \mathbb{P}(U \leq x) &= \sum_{(u,v) \in \mathbb{Z}^2} \mathbb{E} \left[ |G(u,v,a)|^\alpha \{ c_1 \mathbf{1}(G(u,v,a) < 0) + c_2 \mathbf{1}(G(u,v,a) > 0) \} \right]. \end{aligned} \quad (3.20)$$

Here, (3.19) follows from definitions of  $U$  and  $S$ . To prove (3.20), we use ([25], Theorem 3.1). Accordingly, it suffices to check that there exists  $\epsilon > 0$  such that

$$\sum_{(u,v) \in \mathbb{Z}^2} \mathbb{E}|G(u,v,a)|^{\alpha+\epsilon} < \infty \quad \text{and} \quad \sum_{s \in \mathbb{Z}^2} \mathbb{E}|G(u,v,a)|^{\alpha-\epsilon} < \infty, \quad \text{for } 0 < \alpha < 2, \alpha \neq 1, \quad (3.21)$$

$$\mathbb{E} \left( \sum_{(u,v) \in \mathbb{Z}^2} |G(u,v,a)|^{\alpha-\epsilon} \right)^{\frac{\alpha+\epsilon}{\alpha-\epsilon}} < \infty, \quad \text{for } \alpha = 1.$$

Since  $T \subset \mathbb{Z}^2$  is a finite set, it suffices to show (3.21) with  $G(u,v,a)$  replaced by  $g(u,v,a)$ . Let  $1 < \alpha < 2$  and  $p = \alpha + \epsilon > \alpha$  in (3.17). Then  $\sum_{(u,v) \in \mathbb{Z}^2} \mathbb{E}|g(u,v,a)|^{\alpha+\epsilon} \leq C\mathbb{E}[q^{-2(\alpha+\epsilon-1)}(1-A)^{-1}] < \infty$  follows from (3.14) and (3.17). Similarly, if  $1 < \alpha < 2$  and  $1 < p = \alpha - \epsilon \in (1, \alpha)$ , then  $\sum_{(u,v) \in \mathbb{Z}^2} \mathbb{E}|g(u,v,a)|^{\alpha-\epsilon} \leq C\mathbb{E}[q^{-2(\alpha-\epsilon-1)}(1-A)^{-1}] \leq C\mathbb{E}[q^{-2(\alpha+\epsilon-1)}(1-A)^{-1}] < \infty$ , thus proving (3.21) for  $1 < \alpha < 2$ . In the case  $0 < \alpha < 1$ , relations (3.21) immediately follow from (3.14) and (3.17) with  $p = \alpha \pm \epsilon \in (0, 1)$ . Finally, for  $\alpha = 1$ , (3.21) follows from (3.14) and (3.18).  $\square$

**Remark 3.2** For the 2N, 3N, and 4N models in (1.14), (1.15), and (1.16), we have  $q = 1/2, q = 1/3$ , and  $q = 1/2$ , respectively. Hence, for  $1 < \alpha \leq 2$ , condition (3.17) of Proposition 3.2 for the existence of the aggregated random field  $\{\mathfrak{X}(t, s)\}$  in (1.10) reduces to

$$\mathbb{E}(1-A)^{-1} = \int_{[0,1)} (1-a)^{-1} \Phi(da) < \infty. \quad (3.22)$$

For regularly varying mixing density as in (1.13), condition (3.22) is equivalent to  $\beta > 0$ . In the Gaussian case  $\alpha = 2$  the spectral density  $f$  of (1.9) is given in (1.12). For the 2N, 3N, and 4N models we have that  $f(x, y) = \frac{\sigma^2}{(2\pi)^2} \int_{[0,1)} \frac{1}{|1-ae^{i\theta(x,y)}|^2} \Phi(da)$  and hence  $f(x, y)$  is bounded at the origin if and only if

$$f(0, 0) = (\sigma/2\pi)^2 \mathbb{E}(1-A)^{-2} < \infty. \quad (3.23)$$

In particular, for  $\Phi$  as in (1.13) and any  $0 < \beta \leq 1$ , the spectral density  $f$  of the aggregated random field is unbounded.

## 4 Aggregation of the 2N model

According to Proposition 3.2, the aggregated random field of the 2N model in (1.14) is written as

$$\mathfrak{X}_2(t, s) = \sum_{(u,v) \in \mathbb{Z}^2} \int_0^1 g_2(t-u, s-v, a) M_{u,v}(da), \quad (t, s) \in \mathbb{Z}^2, \quad (4.1)$$

where  $\{M_{u,v}(da), (u, v) \in \mathbb{Z}^2\}$  are i.i.d. copies of  $\alpha$ -stable random measure  $M$  on  $(0, 1)$  with control measure  $\Phi(da) = \mathbb{P}(A \in da)$  and the characteristic function  $\mathbb{E}e^{i\theta M(B)} = e^{-|\theta|^\alpha \omega(\theta) \Phi(B)}$ ,  $B \subset (0, 1)$ , see (3.15), (3.16). For  $1 < \alpha \leq 2$ , (4.1) is well-defined, provided the mixing distribution satisfies (3.22); see Remark 3.2.

Next we discuss the asymptotics of the 2N Green function  $g_2$ . It is convenient to change the coordinates as follows:

$$u = t + s, \quad v = t - s, \quad (t, s) \in \mathbb{Z}^2. \quad (4.2)$$

Points  $(u, v) \in \mathbb{Z}^2$  with the property that the sum  $u + v = 2t$  is *even* form the sublattice:

$$\tilde{\mathbb{Z}}^2 := \{(u, v) \in \mathbb{Z}^2 : u + v \text{ is even}\} = \{(u, v) \in \mathbb{Z}^2 : u \equiv v \pmod{2}\}. \quad (4.3)$$

Given a function  $f = f(t, s)$  on  $\mathbb{Z}^2$ , let  $\tilde{f}(u, v) := f((u+v)/2, (u-v)/2)$ ,  $(u, v) \in \tilde{\mathbb{Z}}^2$  denote the corresponding function on  $\tilde{\mathbb{Z}}^2$ . The aggregated random field of (4.1) and the Green function  $g_2$  in the new coordinate system are written as

$$\tilde{\mathfrak{X}}_2(t, s) = \sum_{(u,v) \in \tilde{\mathbb{Z}}^2} \int_0^1 \tilde{g}_2(t-u, s-v, a) \tilde{M}_{u,v}(da), \quad (t, s) \in \tilde{\mathbb{Z}}^2, \quad (4.4)$$

where  $\{\tilde{M}_{u,v}, (u, v) \in \tilde{\mathbb{Z}}^2\}$  are i.i.d. copies of  $M$  and

$$\tilde{g}_2(u, v, a) = a^u p(u, v), \quad (u, v) \in \tilde{\mathbb{Z}}^2, \quad u \geq 0, \quad |v| \leq |u|, \quad (4.5)$$

where

$$\begin{aligned} p(u, v) &:= \frac{1}{2^u} \frac{u!}{\left(\frac{u+v}{2}\right)! \left(\frac{u-v}{2}\right)!}, & \text{if } (u, v) \in \tilde{\mathbb{Z}}^2, u \geq 0, |v| \leq u, \\ &:= 0, & \text{otherwise} \end{aligned} \quad (4.6)$$

is the binomial probability. In other words, the Green function of (1.14) is

$$\begin{aligned} g_2(t, s, a) &= a^{t+s} p(t+s, t-s), & (t, s) \in \mathbb{Z}^2, \quad t+s \geq 0, \quad |t-s| \leq t+s, \\ &= 0, & \text{otherwise.} \end{aligned} \quad (4.7)$$

**Lemma 4.1** *For any  $(t, s, z) \in \mathbb{R}^3$ ,  $t > 0$ ,  $z > 0$ , the point-wise convergence in (1.18) holds. This convergence is uniform on any relatively compact set  $\{\epsilon < t < 1/\epsilon, \epsilon < |s| < 1/\epsilon, \epsilon < z < 1/\epsilon\} \subset \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ ,  $\epsilon > 0$ .*

*Moreover, there exist constants  $C, c > 0$  such that for all sufficiently large  $\lambda$  and any  $(t, s, z)$ ,  $t > 0$ ,  $s \in \mathbb{R}$ ,  $0 < z < \lambda$ ,  $[\lambda t] \stackrel{\text{mod } 2}{=} [\sqrt{\lambda} s]$  the following inequality holds:*

$$\sqrt{\lambda} g_2\left(\frac{[\lambda t] + [\sqrt{\lambda} s]}{2}, \frac{[\lambda t] - [\sqrt{\lambda} s]}{2}, 1 - \frac{z}{\lambda}\right) < C(\bar{h}_2(t, s, z) + \sqrt{\lambda} e^{-zt - c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}), \quad (4.8)$$

where  $\bar{h}_2(t, s, z) := t^{-1/2} e^{-zt - s^2/8t}$ .

The proof of Lemma 4.1 is given in Section 7 (Appendix).

**Theorem 4.1** *Let  $\varepsilon \in D(\alpha)$ ,  $1 < \alpha \leq 2$ . Assume that the mixing density  $\phi$  is bounded on  $(0, 1)$  and satisfies (1.13), where*

$$0 < \beta < \alpha - 1. \quad (4.9)$$

*Let  $\tilde{\mathfrak{X}}_2$  be the aggregated random field in (4.4). Then*

$$n^{-H} \sum_{\substack{1 \leq t \leq [nx], 1 \leq s \leq [\sqrt{n}y] \\ (t, s) \in \tilde{\mathbb{Z}}^2}} \tilde{\mathfrak{X}}_2(t, s) \xrightarrow{\text{fdd}} L_2(x, y), \quad x, y > 0, \quad (4.10)$$

where  $H := \frac{\frac{1}{2} + \alpha - \beta}{\alpha}$ ,

$$L_2(x, y) := (1/2) \int_{\mathbb{R}^2 \times \mathbb{R}_+} \left\{ \int_0^x \int_0^y h_2(t-u, s-v, z) dt ds \right\} \mathcal{M}(du, dv, dz) \quad (4.11)$$

and where  $\mathcal{M}$  is  $\alpha$ -stable random measure on  $\mathbb{R}^2 \times \mathbb{R}_+$  with control measure  $d\mu(u, v, z) = \phi_1 z^\beta du dv dz$  and characteristic function  $\mathbb{E} e^{i\theta \mathcal{M}(B)} = e^{-|\theta|^\alpha \omega(\theta) \mu(B)}$ , where  $B \subset \mathbb{R}^2 \times \mathbb{R}_+$  is a measurable set with  $\mu(B) < \infty$ .

*Proof.* Write  $S_n(x, y)$  for the l.h.s. of (4.10). We prove the convergence of one-dimensional distributions in (4.10) at  $x = y = 1$  only, since the general case of (4.10) is completely analogous. We have

$$\begin{aligned} \mathbb{E} e^{i\theta L_2(1,1)} &= \exp \left\{ -|\theta|^\alpha \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G(u, v, z))^\alpha \omega(\theta G(u, v, z)) d\mu(u, v, z) \right\}, \\ \mathbb{E} e^{i\theta S_n(1,1)} &= \exp \left\{ -|\theta|^\alpha n^{-H\alpha} \sum_{(u,v) \in \tilde{\mathbb{Z}}^2} \mathbb{E} [G_n^\alpha(u, v, A) \omega(\theta G_n(u, v, A))] \right\}, \quad \theta \in \mathbb{R}, \end{aligned}$$

where

$$G(u, v, z) := (1/2) \int_0^1 \int_0^1 h_2(t-u, s-v, z) dt ds, \quad \mathcal{G}_n(u, v, a) := \sum_{1 \leq t \leq n, 1 \leq s \leq [\sqrt{n}]} \tilde{g}_2(t-u, s-v, a). \quad (4.12)$$

Since  $\omega(\theta)$  in (3.15) depends on the sign of  $\theta$  only and  $G \geq 0$ ,  $\mathcal{G}_n \geq 0$ , in the rest of the proof we can assume  $\omega(\cdot) \equiv 1$  without loss of generality, c.f. ([38], proof of Theorem 3.1). Hence, it suffices to show

$$J_n := n^{-H\alpha} \sum_{(u,v) \in \tilde{\mathbb{Z}}^2} \mathbb{E} (\mathcal{G}_n(u, v, A))^\alpha \rightarrow \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G(u, v, z))^\alpha d\mu =: J. \quad (4.13)$$

Let us first check that  $J < \infty$ . More explicitly

$$J = C \int_{\mathbb{R}^2 \times \mathbb{R}_+} \left( \int_0^1 \int_0^1 \frac{1}{\sqrt{(t-u)}} e^{-(s-v)^2/2(t-u)} e^{-z(t-u)} \mathbf{1}(u < t) dt ds \right)^\alpha z^\beta du dv dz = C(J_1 + J_2),$$

where, by Minkowski's inequality,

$$\begin{aligned} J_1 &:= \int_0^\infty du \int_{\mathbb{R}} dv \int_0^\infty z^\beta dz \left( \int_0^1 \int_0^1 \frac{1}{\sqrt{(t+u)}} e^{-(s-v)^2/2(t+u)} e^{-z(t+u)} dt ds \right)^\alpha \\ &\leq \left\{ \int_0^1 \int_0^1 dt ds \left( \int_0^\infty du \int_{\mathbb{R}} dv \int_0^\infty z^\beta dz \frac{1}{(t+u)^{\alpha/2}} e^{-\alpha(s-v)^2/2(t+u)} e^{-\alpha z(t+u)} \right)^{1/\alpha} \right\}^\alpha \\ &= C \left\{ \int_0^1 dt \left( \int_0^\infty du \int_0^\infty z^\beta dz \frac{1}{(t+u)^{\frac{\alpha-1}{2}}} e^{-\alpha z(t+u)} \right)^{1/\alpha} \right\}^\alpha \\ &= C \left\{ \int_0^1 dt \left( \int_0^\infty \frac{du}{(t+u)^{\frac{\alpha-1}{2}+1+\beta}} \right)^{1/\alpha} \right\}^\alpha \\ &= C \left\{ \int_0^1 dt \left( \frac{1}{t^{\frac{\alpha-1}{2}+\beta}} \right)^{1/\alpha} \right\}^\alpha < \infty \end{aligned}$$

since  $\frac{\frac{\alpha-1}{2}+\beta}{\alpha} < 1$  holds because of (4.9) and  $\alpha < 3$ . Next,

$$\begin{aligned} J_2 &:= \int_0^1 dy \int_{\mathbb{R}} dv \int_0^\infty z^\beta dz \left\{ \int_0^1 ds \int_0^y \frac{1}{\sqrt{x}} e^{-(s-v)^2/2x} e^{-zx} dx \right\}^\alpha \\ &= \int_0^1 dy \int_{|v| \leq 2} dv \int_0^\infty z^\beta dz \{ \dots \}^\alpha + \int_0^1 dy \int_{|v| > 2} dv \int_0^\infty z^\beta dz \{ \dots \}^\alpha =: J_{21} + J_{22}. \end{aligned}$$

Here,

$$J_{21} \leq C \int_0^\infty z^\beta dz \left\{ \int_0^1 e^{-zx} dx \right\}^\alpha = C \int_0^\infty z^{\beta-\alpha} (1 - e^{-z})^\alpha dz < \infty$$

since  $\alpha > 1 + \beta$ . Finally, since  $(s-v)^2 \geq v^2/4$  for  $|s| < 1$ ,  $|v| > 2$ , so  $\int_0^1 e^{-(s-v)^2/2x} ds \leq e^{-v^2/8x} \leq C(x/v^2)$  ( $|v| > 2, 0 < x < 1$ ) and

$$\begin{aligned} J_{22} &\leq C \int_{|v| > 2} |v|^{-2\alpha} dv \int_0^\infty z^\beta dz \left\{ \int_0^1 x^{1/2} e^{-zx} dx \right\}^\alpha \\ &\leq C \left\{ \int_0^1 x^{1/2} dx \left( \int_0^\infty e^{-\alpha zx} z^\beta dz \right)^{1/\alpha} \right\}^\alpha = C \left\{ \int_0^1 \frac{x^{1/2} dx}{x^{\frac{1+\beta}{\alpha}}} \right\}^\alpha < \infty, \end{aligned}$$

since  $-\frac{1}{2} + \frac{1+\beta}{\alpha} < 1$ . This proves  $J < \infty$ , or  $G \in L^\alpha(\mu)$ .

Let us prove the convergence in (4.13). For notational simplicity we can assume  $\phi(a) = (1-a)^\beta$ , c.f. [38]. Then

$$J_n = \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_n(u, v, z))^\alpha d\mu(u, v, z),$$

where

$$G_n(u, v, z) := \int_{(0,1]^2} \sqrt{n} \tilde{g}_2([nt] - [nu], [\sqrt{n}s] - [\sqrt{n}v], 1 - \frac{z}{n}) \mathbf{1}(0 < z < n) \mathbf{1}(B_n(t, s, u, v)) dt ds$$

and  $\mathbf{1}(B_n(t, s, u, v)) := \mathbf{1}([nt] - [nu] \stackrel{\text{mod } 2}{=} [\sqrt{n}s] - [\sqrt{n}v])$ . Let  $W_\epsilon := \{(u, v, z) \in \mathbb{R}^2 \times \mathbb{R}_+ : |u|, |v| < 1/\epsilon, \epsilon < z < 1/\epsilon\}$ . We claim that

$$\lim_{n \rightarrow \infty} \sup_{(u, v, z) \in W_\epsilon} |G_n(u, v, z) - G(u, v, z)| = 0, \quad \forall \epsilon > 0. \quad (4.14)$$

To show (4.14), for given  $\epsilon_1 > 0$  split  $G_n(u, v, z) - G(u, v, z) = \sum_{j=1}^4 \Gamma_{nj}(u, v, z)$ , where, for  $0 < z < n$ ,

$$\begin{aligned}\Gamma_{n1}(u, v, z) &:= \int_{(0,1]^2 \cap D(\epsilon_1)} \{ \sqrt{n} \tilde{g}_2([nt] - [nu], [\sqrt{n}s] - [\sqrt{n}v], 1 - \frac{z}{n}) - h_2(t - u, s - v, z) \} \mathbf{1}(B_n(t, s, u, v)) dt ds, \\ \Gamma_{n2}(u, v, z) &:= \int_{(0,1]^2 \cap D(\epsilon_1)} (h_2(t - u, s - v, z) \mathbf{1}(B_n(t, s, u, v))) dt ds - (1/2) \int_{(0,1]^2 \cap D(\epsilon_1)} h_2(t - u, s - v, z) dt ds, \\ \Gamma_{n3}(u, v, z) &:= \int_{(0,1]^2 \cap D(\epsilon_1)^c} \sqrt{n} \tilde{g}_2([nt] - [nu], [\sqrt{n}s] - [\sqrt{n}v], 1 - \frac{z}{n}) \mathbf{1}(B_n(t, s, u, v)) dt ds, \\ \Gamma_{n4}(u, v, z) &:= -\frac{1}{2} \int_{(0,1]^2 \cap D(\epsilon_1)^c} h_2(t - u, s - v, z) dt ds,\end{aligned}$$

and where the sets  $D(\epsilon), D(\epsilon)^c$  (depending on  $u, v$ ) are defined by

$$D(\epsilon) := \{(t, s) \in (0, 1]^2 : t - u > \epsilon, |s - v| > \epsilon\}, \quad D(\epsilon)^c := (0, 1]^2 \setminus D(\epsilon).$$

To show (4.14), it suffices to verify that for any  $\epsilon > 0, \delta > 0$  there exists  $\epsilon_1 > 0, n_1 \geq 1$  such that

$$\lim_{n \rightarrow \infty} \sup_{(u, v, z) \in W_\epsilon} \Gamma_{ni}(u, v, z) = 0, \quad i = 1, 2, \quad (4.15)$$

$$\sup_{(u, v, z) \in W_\epsilon} |\Gamma_{ni}(u, v, z)| < \delta, \quad i = 3, 4, \quad \forall n \geq n_1. \quad (4.16)$$

Relation (4.15) for  $i = 1$  follows from Lemma 4.1 or (7.1), and for  $i = 2$  it follows from uniform continuity of  $h_2$  on compact subsets of  $\mathbb{R}_0^2 \times \mathbb{R}_+$ .

Next,  $|\Gamma_{n4}(u, v, z)| \leq C \int_0^{\epsilon_1} t^{-1/2} dt + C \int_{\epsilon_1}^1 t^{-1/2} dt \int_{|s| < \epsilon_1} ds = O(\sqrt{\epsilon_1})$ , implying (4.16) for  $i = 4$  with  $\epsilon_1 = C\delta^2$ . Finally, using (4.8) we obtain  $|\Gamma_{n3}(u, v, z)| \leq C\sqrt{\epsilon_1} + C\sqrt{n} \int_0^1 e^{-c(nt)^{1/3}} dt \leq C\sqrt{\epsilon_1} + C/\sqrt{n} < \delta$  provided  $\sqrt{\epsilon_1} < \delta/(2C), n > n_1 = (2C/\delta)^2$  hold. This proves (4.16) for  $i = 3$  and hence (4.14), too.

Let

$$G'_n(u, v, z) := \sqrt{n} \mathbf{1}(0 < z < n) \int_{(0,1]^2} e^{-z(t-u) - c(n(t-u))^{1/3} - c(\sqrt{n}|s-v|)^{1/2}} \mathbf{1}(t > u) dt ds, \quad (4.17)$$

where  $c > 0$  is the same as in (4.8) and (7.2). Let us show that

$$J'_n := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G'_n(u, v, z))^\alpha d\mu = o(1). \quad (4.18)$$

Split  $J'_n = \sum_{i=1}^3 I_{ni}$ , where

$$I_{n1} := \int_{(-\infty, 0] \times \mathbb{R}_+ \times \mathbb{R}_+} (G'_n)^\alpha d\mu, \quad I_{n2} := \int_{(0,1] \times [-2,2] \times \mathbb{R}_+} (G'_n)^\alpha d\mu, \quad I_{n3} := \int_{(0,1] \times [-2,2]^c \times \mathbb{R}_+} (G'_n)^\alpha d\mu,$$

$[-2, 2]^c = \mathbb{R} \setminus [-2, 2]$ . Using the fact that  $\int_{\mathbb{R}} e^{-cn^{1/4}|s-v|^{1/2}} dv = C/\sqrt{n}$  and Minkowski's inequality,

$$\begin{aligned}I_{n1} &\leq Cn^{\alpha/2} \left\{ \int_{(0,1]^2} dt ds \left( \int_{\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+} e^{-\alpha z(t+u) - c\alpha(n(t+u))^{1/3} - c\alpha(\sqrt{n}|s-v|)^{1/2}} z^\beta du dv dz \right)^{1/\alpha} \right\}^\alpha \\ &\leq Cn^{\frac{\alpha-1}{2}} \left\{ \int_0^1 dt \left( \int_0^\infty e^{-c\alpha(n(t+u))^{1/3}} \frac{du}{(t+u)^{1+\beta}} \right)^{1/\alpha} \right\}^\alpha \leq Cn^{-(\frac{\alpha+1}{2}-\beta)} I,\end{aligned}$$

where  $\frac{\alpha+1}{2} - \beta > 0$  and  $I := \left\{ \int_0^\infty dt \left( \int_0^\infty e^{-c\alpha(n(t+u))^{1/3}} (t+u)^{-1-\beta} du \right)^{1/\alpha} \right\}^\alpha < \infty$ . Next,

$$\begin{aligned}I_{n2} &\leq Cn^{\alpha/2} \int_0^\infty z^\beta dz \left\{ \int_{(0,4]^2} e^{-zt - c(nt)^{1/3} - c(\sqrt{n}|s|)^{1/2}} dt ds \right\}^\alpha \\ &\leq C \left\{ \int_0^4 e^{-c(nt)^{1/3}} dt \left( \int_0^\infty e^{-\alpha zt} z^\beta dz \right)^{1/\alpha} \right\}^\alpha \\ &\leq C \left\{ \int_0^\infty e^{-c(nt)^{1/3}} t^{-\frac{1+\beta}{\alpha}} dt \right\}^\alpha \leq Cn^{-(\alpha-1-\beta)} = o(1).\end{aligned}$$

Finally, using  $e^{-c(\sqrt{n}|s-v|)^{1/2}} \leq e^{-(c/2)(\sqrt{n}|v|)^{1/2}}$  for  $|v| \geq 2, |s| \leq 1$  it easily follows  $I_{n3} = O(e^{-c'n^{1/4}}) = o(1)$  ( $\exists c' > 0$ ), thus completing the proof of (4.18).

With (4.14) and (4.18) in mind, write

$$\begin{aligned} |J_n - J| &\leq \int_{W_\epsilon} |G_n^\alpha - G^\alpha| d\mu + \int_{W_\epsilon^c} |G_n|^\alpha d\mu + \int_{W_\epsilon^c} |G|^\alpha d\mu \\ &\leq \int_{W_\epsilon} |G_n^\alpha - G^\alpha| d\mu + C \int_{\mathbb{R}^2 \times \mathbb{R}_+} |G'_n|^\alpha d\mu + C \int_{W_\epsilon^c} |\bar{G}|^\alpha d\mu + \int_{W_\epsilon^c} |G|^\alpha d\mu, \end{aligned} \quad (4.19)$$

where  $\bar{G}(u, v, z) := \int_0^1 \int_0^1 \bar{h}_2(t - u, s - v, z) dt ds$ ,  $W_\epsilon^c := \mathbb{R}^2 \times \mathbb{R}_+ \setminus W_\epsilon$ . Since  $G, \bar{G} \in L^\alpha(\mu)$ , the third and fourth terms on the r.h.s. of (4.19) can be made arbitrary small by choosing  $\epsilon > 0$  small enough. Next, for a given  $\epsilon > 0$ , the first term on the r.h.s. of (4.19) vanishes in view of (4.14), and the second term tends to zero, see (4.18). This proves (4.13), thus concluding the proof Theorem 4.1.

**Theorem 4.2** *Assume conditions and notation of Theorem 4.1. Then*

$$n^{-H_\star} \sum_{\substack{1 \leq t \leq [nx], 1 \leq s \leq [ny] \\ (t, s) \in \tilde{\mathbb{Z}}^2}} \tilde{\mathfrak{X}}_2(t, s) \xrightarrow{\text{fdd}} L_{2\star}(x, y), \quad x, y > 0, \quad (4.20)$$

where  $H_\star := \frac{1+\alpha-\beta}{\alpha}$ ,

$$\begin{aligned} L_{2\star}(x, y) &:= (1/2) \int_{\mathbb{R}^2 \times \mathbb{R}_+} \mathcal{M}(du, dv, dz) \mathbf{1}(0 < v < y) \int_0^x h_{2\star}(t - u, z) dt, \\ h_{2\star}(t, z) &:= \int_{\mathbb{R}} h_2(t, s, z) ds = 2e^{-tz} \mathbf{1}(t > 0) \end{aligned} \quad (4.21)$$

and where  $\mathcal{M}$  is the same  $\alpha$ -stable random measure as in Theorem 4.1.

*Proof.* Similarly as in the case of Theorem 4.1, we prove one-dimensional convergence in (4.20) at  $x = y = 1$  only, and assume  $\Phi(da) = (1 - a)^\beta da$ . Correspondingly, it suffices to show the limit  $\lim J_{n\star} = J_\star$ , where

$$J_{n\star} := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_{n\star}(u, v, z))^\alpha d\mu(u, v, z), \quad J_\star := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_\star(u, v, z))^\alpha d\mu(u, v, z),$$

where

$$\begin{aligned} G_\star(u, v, z) &:= (1/2) \mathbf{1}(0 < v < 1) \int_0^1 h_{2\star}(t - u, z) dt, \\ G_{n\star}(u, v, z) &:= \int_0^1 dt \sum_{s=1}^n \tilde{g}_2([nt] - [nu], s - [nv], 1 - \frac{z}{n}) \mathbf{1}(B_{n\star}(t, s, u, v)) \mathbf{1}(0 < z < n) \end{aligned}$$

and  $\mathbf{1}(B_{n\star}(t, s, u, v)) := \mathbf{1}([nt] - [nu] \stackrel{\text{mod } 2}{=} s - [nv])$ .

Define  $J'_{n\star} := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_{n\star}(u, v, z))^\alpha \mathbf{1}(|v| \leq 3) d\mu$ ,  $J''_{n\star} := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_{n\star}(u, v, z))^\alpha \mathbf{1}(|v| > 3) d\mu$ ,  $J'_{n\star} + J''_{n\star} = J_{n\star}$ . Then  $\lim J_{n\star} = J_\star$  follows from  $\lim J'_{n\star} = J_\star$  and  $\lim J''_{n\star} = 0$ .

Note that for any  $u < t$

$$\sum_{s=1}^n p([nt] - [nu], s - [nv]) \rightarrow \begin{cases} 1, & v \in (0, 1), \\ 0, & v > 1 \text{ or } v < 0. \end{cases}$$

Hence and from the proof of Lemma 4.1 it easily follows that

$$\lim_{n \rightarrow \infty} G_{n\star}(u, v, z) = G_\star(u, v, z), \quad \forall u \in \mathbb{R}, v \in \mathbb{R} \setminus \{0, 1\}, z > 0. \quad (4.22)$$

Also note

$$G_{n\star}(u, v, z) \leq \int_0^1 \left(1 - \frac{z}{n}\right)^{[nt] - [nu]} dt \mathbf{1}(u < t, 0 < z < n) \leq C \int_0^1 h_{2\star}(t - u, z) dt, \quad \forall (u, v, z) \in \mathbb{R}^2 \times \mathbb{R}_+. \quad (4.23)$$

Relations (4.22)-(4.23) and the dominated convergence theorem entail the convergence  $\lim J'_{n\star} = J_\star$ .

It remains to evaluate the remainder term  $J''_{n\star}$ , which can be rewritten as

$$J''_{n\star} := \frac{1}{n^{\alpha H_\star}} \sum_{(u,v) \in \mathbb{Z}^2: |v| > 3n} \mathbb{E} \left( \sum_{t,s=1}^n A^{t-u} p(t-u, s-v) \right)^\alpha.$$

Observe  $|v| > 3n$ ,  $u \leq t$ ,  $|s-v| \leq t-u$ ,  $1 \leq s, t \leq n$  imply  $u < -n$ . Split  $J''_{n\star} = n^{-\alpha H_\star} (T_{n1} + T_{n2})$ , where  $T_{n1} := \sum_{u < -n, v > 3n} \dots$ ,  $T_{n2} := \sum_{u < -n, v < -3n} \dots$ . Inequality  $\phi(a) \leq C(1-a)^\beta$ ,  $a \in (0, 1)$  implies  $\mathbb{E} A^t \leq C \int_0^1 z^\beta (1-z)^t dz \leq C \int_0^\infty z^\beta e^{-zt} dz = Ct^{-1-\beta}$ ,  $t \geq 1$ . Using the above fact and Minkowski's inequality, we obtain

$$\begin{aligned} T_{n1} &\leq C \left\{ \sum_{t,s=1}^n \left( \sum_{u > n, v > n} \mathbb{E}[A^{\alpha(t+u)}] p^\alpha(t+u, v+s) \right)^{1/\alpha} \right\}^\alpha \\ &\leq C n^{2\alpha} \sum_{u > n, v > n} \mathbb{E}[A^{\alpha u}] p^\alpha(u, v) \\ &\leq C n^{2\alpha} \sum_{u > n} \frac{1}{u^{1+\beta}} \sum_{v > n} p^\alpha(u, v) =: C n^{2\alpha} I_n. \end{aligned}$$

For sufficiently large  $K > 0$ , split  $I_n = I_{n1} + I_{n2}$ , where

$$I_{n1} := \sum_{u > n} \frac{1}{u^{1+\beta}} \sum_{v > n} p^\alpha(u, v) \mathbf{1}(u^2 > K v^3), \quad I_{n2} := \sum_{u > n} \frac{1}{u^{1+\beta}} \sum_{v > n} p^\alpha(u, v) \mathbf{1}(u^2 \leq K v^3).$$

From the Moivre-Laplace approximation in (7.6),

$$\begin{aligned} I_{n1} &\leq C \sum_{u > n} \frac{1}{u^{1+\beta+\frac{\alpha}{2}}} \sum_{v > n} e^{-\alpha v^2/2u} \\ &\leq C \sum_{u > n} \frac{1}{u^{1+\beta+\frac{\alpha-1}{2}}} e^{-\alpha n^2/2u} = C \left( \sum_{n < u < n^2} \dots + \sum_{u \geq n^2} \dots \right) =: C(I'_{n1} + I''_{n1}). \end{aligned}$$

Here,  $I''_{n1} \leq C \sum_{u \geq n^2} u^{-1-\beta-\frac{\alpha-1}{2}} \leq C/n^{2\beta+\alpha-1}$ . A similar bound for  $I'_{n1}$  follows by integral approximation:

$$I'_{n1} \leq \frac{C}{n^{2\beta+\alpha-1}} \int_0^1 \frac{dx}{x^{1+\beta+\frac{\alpha-1}{2}}} e^{-\alpha/2x} \leq \frac{C}{n^{2\beta+\alpha-1}}$$

since the last integral converges. Therefore,  $I_{n1} = O(n^{-2\beta-\alpha+1})$  and hence  $n^{2\alpha} I_{n1} = o(n^{\alpha H_\star})$  by the definition of  $H_\star$  and condition  $\beta > 0$ .

Next, using Hoeffding's inequality in (7.10), for some constant  $c > 0$  we obtain

$$I_{n2} \leq C \sum_{u > n} \frac{1}{u^{1+\beta}} e^{-c \max(u^{1/3}, n^2/u)} \leq C \sum_{n < u \leq n^{3/2}} \frac{1}{u^{1+\beta}} e^{-cn^{1/2}} + C \sum_{u \geq n^{3/2}} \frac{1}{u^{1+\beta}} e^{-cu^{1/3}} = O(e^{-cn^{1/2}}).$$

Hence,  $T_{n1} = o(n^{\alpha H_\star})$ . Since estimation of  $T_{n2}$  is completely analogous, this proves  $J''_{n\star} = o(1)$  and Theorem 4.2, too.

**Proposition 4.2** *Let  $\{L_2(x, y)\}$ ,  $\{L_{2\star}(x, y)\}$  be the random fields in (4.11), (4.21), respectively, and (4.9) holds. Then, as  $\lambda \rightarrow \infty$ ,*

$$\lambda^{-1/\alpha} L_2(x, \lambda y) \xrightarrow{\text{fdd}} L_{2\star}(x, y), \quad x, y > 0. \quad (4.24)$$

*Proof.* We restrict the proof of (4.24) to that of the one-dimensional convergence at  $x = y = 1$ . Accordingly, it suffices to show  $\lim_{\lambda \rightarrow \infty} J_\lambda = J$ , where

$$J_\lambda := \frac{1}{\lambda} \int \left( \int_0^1 \int_0^\lambda h_2(t-u, s-v, z) dt ds \right)^\alpha d\mu, \quad J := \int \left( \int_0^1 h_{2\star}(t-u, z) dt \mathbf{1}(0 < v < 1) \right)^\alpha d\mu$$



and  $\int = \int_{\mathbb{R}^2 \times \mathbb{R}_+}$ . Split  $J_\lambda = J'_\lambda + J''_\lambda$ , where  $J'_\lambda := \lambda^{-1} \int \mathbf{1}(|v| < 2\lambda)(\cdots)^\alpha d\mu$ ,  $J''_\lambda := \lambda^{-1} \int \mathbf{1}(|v| \geq 2\lambda)(\cdots)^\alpha d\mu$ . The limit  $J'_\lambda \rightarrow J$  follows by the dominated convergence theorem and the change of variable  $v \rightarrow \lambda v$  using the facts that

$$q_\lambda(t, u, v) := \frac{1}{\sqrt{2\pi(t-u)}} \int_0^\lambda e^{-(s-\lambda v)^2/2(t-u)} ds \rightarrow \begin{cases} 1, & v \in (0, 1), \\ 0, & v \notin [0, 1], \end{cases}$$

and  $|q_\lambda(t, u, v)| \leq 1$  for any  $t > u, v \in \mathbb{R}$ . The proof of  $J''_\lambda \rightarrow 0$  is similar to the proof of  $J''_{n*} \rightarrow 0$  in Theorem 4.2 and is omitted. Proposition 4.2 is proved.

**Remark 4.1** [38] discussed aggregation of stationary AR(1) processes  $X(t) = AX(t-1) + \varepsilon(t)$ ,  $t \in \mathbb{Z}$  with random coefficient  $A \in (0, 1)$  and i.i.d. innovations  $\{\varepsilon(t)\} \in D(\alpha)$ ,  $0 < \alpha \leq 2$ . Under similar assumptions as in Theorem 4.1, they proved that normalized partial sums of the corresponding aggregated  $\alpha$ -stable process  $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$  tend, in distribution, to a self-similar process  $\{L(x), x > 0\}$  written as the stochastic integral

$$L(x) = \int_{\mathbb{R} \times \mathbb{R}_+} (f(x-u, z) - f(-u, z)) \nu(du, dz), \quad (4.25)$$

where  $f(u, z) := (1 - e^{-zu})\mathbf{1}(u > 0)$  and  $\nu$  is an  $\alpha$ -stable random measure on  $\mathbb{R} \times \mathbb{R}_+$  with control measure  $Cz^{\beta-\alpha} du dz$ . Note that  $f(x-u, z) - f(-u, z) = (z/2) \int_0^x h_{2*}(t-u, z) dt$ , where  $h_{2*}$  is defined in (4.21). Hence, (4.25) can be rewritten as  $L(x) = \int_{\mathbb{R} \times \mathbb{R}_+} \tilde{\nu}(du, dz) \int_0^x h_{2*}(t-u, z) dt$ , where  $\tilde{\nu}(du, dz) := (z/2)\nu(du, dz)$  is  $\alpha$ -stable random measure with control measure  $Cz^\beta du dz$ . The above facts imply that for any fixed  $y > 0$ , the sectional random process  $\{L_{2*}(x, y), x > 0\}$  in (4.21) coincides (up to a multiplicative constant) with the process  $\{L(x), x > 0\}$  of (4.25), in distribution. Note that for  $0 < y_1 < y_2$ , the increment process  $\{L_{2*}(x, y_2) - L_{2*}(x, y_1), x > 0\}$  is independent of  $\{L_{2*}(x, y_1), x > 0\}$  and  $\{L_{2*}(x, y_2) - L_{2*}(x, y_1), x > 0\} \stackrel{\text{fdd}}{=} \{L_{2*}(x, y_2 - y_1), x > 0\}$ .

**Proposition 4.1** *Let the conditions of Theorem 4.1 be satisfied. Then:*

- (i) *The random field  $\{\mathfrak{X}_2(t, s)\}$  in (4.1) has anisotropic distributional long memory with parameters  $H_1 = H = \frac{\frac{1}{2} + \alpha - \beta}{\alpha}$ ,  $H_2 = 2H_1$ .*
- (ii) *The random field  $\{\mathfrak{X}_2(t, s)\}$  in (4.1) does not have isotropic distributional long memory.*

The proof of Proposition 4.1 is analog to the proof of Proposition 5.1.

## 5 Aggregation of the 3N model

In this section we prove the anisotropic long memory properties, in the sense of Definition 2.2 of Section 2, of the aggregated 3N model given by

$$\mathfrak{X}_3(t, s) = \sum_{(u, v) \in \mathbb{Z}^2} \int_0^1 g_3(t-u, s-v, a) M_{u, v}(da), \quad (t, s) \in \mathbb{Z}^2, \quad (5.1)$$

where  $\{M_{u, v}(da), (u, v) \in \mathbb{Z}^2\}$  are i.i.d. copies of  $\alpha$ -stable random measure  $M$  on  $[0, 1)$  with control measure  $\Phi(da) = P(A \in da)$  and the characteristic function  $\mathbb{E} e^{i\theta M(B)} = e^{-|\theta|^\alpha \omega(\theta) \Phi(B)}$ ,  $B \subset [0, 1)$ , see (3.15), (3.16), and  $g_3(t, s, a)$  is the Green function of the random walk  $\{W_k\}$  on  $\mathbb{Z}^2$  with one-step transition probabilities shown in Figure 1 b). For  $1 < \alpha \leq 2$ , (5.1) is well-defined, provided the mixing distribution satisfies (3.22).

Introduce a random field  $\{V_3(x, y), (x, y) \in \mathbb{R}_+^2\}$  as a stochastic integral

$$V_3(x, y) := \int_{\mathbb{R}^2 \times \mathbb{R}_+} \mathcal{M}(du, dv, dz) \int_0^x \int_0^y h_3(t-u, s-v, z) dt ds, \quad (5.2)$$

where  $\mathcal{M}$  is  $\alpha$ -stable random measure on  $\mathbb{R}^2 \times \mathbb{R}_+$  with control measure  $d\mu(u, v, z) := \phi_1 z^\beta du dv dz$  and characteristic function  $\mathbb{E} e^{i\theta \mathcal{M}(B)} = e^{-|\theta|^\alpha \omega(\theta) \mu(B)}$ , where  $B \subset \mathbb{R}^2 \times \mathbb{R}_+$  is a measurable set with  $\mu(B) < \infty$ . As shown in the

proof of Theorem 5.1 below, the stochastic integral in (5.2) is well-defined. The random field in (5.2) has  $\alpha$ -stable finite-dimensional distributions and stationary increments in the sense that for any  $(u, v) \in \mathbb{R}_+^2$

$$\{V_3(x, y), (x, y) \in \mathbb{R}_+^2\} \stackrel{\text{fdd}}{=} \{V_3(u+x, v+y) - V_3(u, v+y) - V_3(u+x, v) + V_3(u, v), (x, y) \in \mathbb{R}_+^2\}. \quad (5.3)$$

Moreover, (5.2) is OSRF and satisfies (1.4), viz.,

$$\{V_3(\lambda x, \sqrt{\lambda} y)\} \stackrel{\text{fdd}}{=} \{\lambda^H V_3(x, y)\}, \quad (5.4)$$

with  $H$  given in (5.7). Property (5.4) is immediate from the scaling properties  $h_3(\lambda u, \sqrt{\lambda} v, \lambda^{-1} z) = \lambda^{-1/2} h_3(u, v, z)$  and  $\{\mathcal{M}(d\lambda u, d\sqrt{\lambda} v, d\lambda^{-1} z)\} \stackrel{\text{fdd}}{=} \{\lambda^{\frac{1}{2}-\beta} \mathcal{M}(du, dv, dz)\}$ , the last property being a consequence of the scaling property of  $\mu(d\lambda u, d\sqrt{\lambda} v, d\lambda^{-1} z) = \lambda^{\frac{1}{2}-\beta} \mu(du, dv, dz)$  of the control measure  $\mu$ .

**Remark 5.1** The random field (5.2) is different from the class of  $\alpha$ -stable OSRFs defined in ([6], (3.1)) because the latter fields satisfy a different stationary increment property, see ([6], (3.5)). Moreover, (5.2) have a mixed moving average representation in contrast to the moving average representation in ([6], (3.1)).

The main result of this section is Theorem 5.1. Its proof is based on the asymptotics of the Green function  $g_3$  in Lemma 5.1, below. The proof of Lemma 5.1 is given in Section 7 (Appendix).

**Lemma 5.1** *For any  $(t, s, z) \in (0, \infty) \times \mathbb{R} \times (0, \infty)$  the point-wise convergence in (1.19) holds. This convergence is uniform on any relatively compact set  $\{\epsilon < t < 1/\epsilon, \epsilon < |s| < 1/\epsilon, \epsilon < z < 1/\epsilon\} \subset (0, \infty) \times \mathbb{R} \times (0, \infty)$ ,  $\epsilon > 0$ . Moreover, there exist constants  $C, c > 0$  such that for all sufficiently large  $\lambda$  and any  $(t, s, z)$ ,  $t > 0$ ,  $s \in \mathbb{R}$ ,  $0 < z < \lambda$  the following inequality holds:*

$$\sqrt{\lambda} g_3([\lambda t], [\sqrt{\lambda} s], 1 - \frac{z}{\lambda}) < C(\bar{h}_3(t, s, z) + \sqrt{\lambda} e^{-zt - c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}), \quad (5.5)$$

where  $\bar{h}_3(t, s, z) := \frac{1}{\sqrt{t}} e^{-zt - \frac{z^2}{16t}}$ ,  $(t, s, z) \in (0, \infty) \times \mathbb{R} \times (0, \infty)$ .

**Theorem 5.1** *Assume that the mixing density  $\phi$  is bounded on  $[0, 1]$  and satisfies (1.13), where*

$$0 < \beta < \alpha - 1, \quad 1 < \alpha \leq 2. \quad (5.6)$$

*Let  $\{\mathfrak{X}_3(t, s)\}$  be the aggregated random field in (5.1). Then*

$$n^{-H} \sum_{t=1}^{[nx]} \sum_{s=1}^{[\sqrt{n}y]} \mathfrak{X}_3(t, s) \stackrel{\text{fdd}}{\rightarrow} V_3(x, y), \quad x, y > 0, \quad H := \frac{\frac{1}{2} + \alpha - \beta}{\alpha}. \quad (5.7)$$

*Proof.* Write  $S_n(x, y)$  for the l.h.s. of (5.7). We prove the convergence of one-dimensional distributions in (5.7) at  $x = y = 1$  only, since the general case of (5.7) is completely analogous. We have

$$\begin{aligned} \mathbb{E} e^{i\theta V_3(1,1)} &= \exp \left\{ -|\theta|^\alpha \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G(u, v, z))^\alpha \omega(\theta G(u, v, z)) d\mu(u, v, z) \right\}, \\ \mathbb{E} e^{i\theta S_n(1,1)} &= \exp \left\{ -|\theta|^\alpha n^{-H\alpha} \sum_{(u,v) \in \mathbb{Z}^2} \mathbb{E} [\mathcal{G}_n^\alpha(u, v, A) \omega(\theta \mathcal{G}_n(u, v, A))] \right\}, \quad \theta \in \mathbb{R}, \end{aligned}$$

where

$$G(u, v, z) := \int_0^1 \int_0^1 h_3(t-u, s-v, z) dt ds, \quad \mathcal{G}_n(u, v, a) := \sum_{1 \leq t \leq n, 1 \leq s \leq [\sqrt{n}]} g_3(t-u, s-v, a). \quad (5.8)$$

Since  $\omega(\theta)$  in (3.15) depends on the sign of  $\theta$  only and  $G \geq 0$ ,  $\mathcal{G}_n \geq 0$ , in the rest of the proof we can assume  $\omega(\cdot) \equiv 1$  without loss of generality, c.f. ([38], proof of Theorem 3.1). Hence, it suffices to show

$$J_n := n^{-H\alpha} \sum_{(u,v) \in \mathbb{Z}^2} \mathbb{E} (\mathcal{G}_n(u, v, A))^\alpha \rightarrow \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G(u, v, z))^\alpha d\mu =: J. \quad (5.9)$$

Let us first check that  $J < \infty$ , i.e., that  $V_3(1, 1)$  is well-defined as a stochastic integral with respect to  $\mathcal{M}$ . We have

$$J = C \int_{\mathbb{R}^2 \times \mathbb{R}_+} \left( \int_0^1 \int_0^1 \frac{1}{\sqrt{(t-u)}} e^{-(s-v)^2/4(t-u)} e^{-3z(t-u)} \mathbf{1}(u < t) dt ds \right)^\alpha z^\beta du dv dz = C(J_1 + J_2),$$

where, by Minkowski's inequality,

$$\begin{aligned} J_1 &:= \int_0^\infty du \int_{\mathbb{R}} dv \int_0^\infty z^\beta dz \left( \int_0^1 \int_0^1 \frac{1}{\sqrt{(t+u)}} e^{-(s-v)^2/4(t+u)} e^{-3z(t+u)} dt ds \right)^\alpha \\ &\leq \left\{ \int_0^1 \int_0^1 dt ds \left( \int_0^\infty du \int_{\mathbb{R}} dv \int_0^\infty z^\beta dz \frac{1}{(t+u)^{\alpha/2}} e^{-\alpha(s-v)^2/4(t+u)} e^{-3\alpha z(t+u)} \right)^{1/\alpha} \right\}^\alpha \\ &= C \left\{ \int_0^1 dt \left( \int_0^\infty du \int_0^\infty z^\beta dz \frac{1}{(t+u)^{\frac{\alpha-1}{2}}} e^{-3\alpha z(t+u)} \right)^{1/\alpha} \right\}^\alpha \\ &= C \left\{ \int_0^1 dt \left( \int_0^\infty \frac{du}{(t+u)^{\frac{\alpha-1}{2}+1+\beta}} \right)^{1/\alpha} \right\}^\alpha \\ &= C \left\{ \int_0^1 dt \left( \frac{1}{t^{\frac{\alpha-1}{2}+\beta}} \right)^{1/\alpha} \right\}^\alpha < \infty \end{aligned}$$

since  $\frac{\frac{\alpha-1}{2}+\beta}{\alpha} < 1$  holds because of (5.6) and  $\alpha < 3$ . Next,

$$\begin{aligned} J_2 &:= \int_0^1 dy \int_{\mathbb{R}} dv \int_0^\infty z^\beta dz \left\{ \int_0^1 ds \int_0^y \frac{1}{\sqrt{x}} e^{-(s-v)^2/4x} e^{-3zx} dx \right\}^\alpha \\ &= \int_0^1 dy \int_{|v| \leq 2} dv \int_0^\infty z^\beta dz \{ \dots \}^\alpha + \int_0^1 dy \int_{|v| > 2} dv \int_0^\infty z^\beta dz \{ \dots \}^\alpha =: J_{21} + J_{22}. \end{aligned}$$

Here,

$$J_{21} \leq C \int_0^\infty z^\beta dz \left\{ \int_0^1 e^{-3zx} dx \right\}^\alpha = C \int_0^\infty z^{\beta-\alpha} (1 - e^{-z})^\alpha dz < \infty$$

since  $\alpha > 1 + \beta$ . Finally, since  $(s-v)^2 \geq v^2/4$  for  $|s| < 1$ ,  $|v| > 2$ , so  $\int_0^1 e^{-(s-v)^2/4x} ds \leq e^{-v^2/16x} \leq C(x/v^2) (|v| > 2, 0 < x < 1)$  and

$$\begin{aligned} J_{22} &\leq C \int_{|v| > 2} |v|^{-2\alpha} dv \int_0^\infty z^\beta dz \left\{ \int_0^1 x^{1/2} e^{-3zx} dx \right\}^\alpha \\ &\leq C \left\{ \int_0^1 x^{1/2} dx \left( \int_0^\infty e^{-3\alpha z x} z^\beta dz \right)^{1/\alpha} \right\}^\alpha = C \left\{ \int_0^1 \frac{x^{1/2} dx}{x^{\frac{1+\beta}{\alpha}}} \right\}^\alpha < \infty, \end{aligned}$$

since  $-\frac{1}{2} + \frac{1+\beta}{\alpha} < 1$ . This proves  $J < \infty$ , or  $G \in L^\alpha(\mu)$ .

Let us prove the convergence in (5.9). For notational simplicity we can assume  $\phi(a) = (1-a)^\beta$ , c.f. ([38], proof of Theorem 3.1). Then

$$J_n = \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_n(u, v, z))^\alpha d\mu(u, v, z),$$

where

$$G_n(u, v, z) := \int_{(0,1]^2} \sqrt{n} g_3([nt] - [nu], [\sqrt{n}s] - [\sqrt{n}v], 1 - \frac{z}{n}) \mathbf{1}(0 < z < n) dt ds.$$

Let  $W_\epsilon := \{(u, v, z) \in \mathbb{R}^2 \times \mathbb{R}_+ : |u|, |v| < 1/\epsilon, \epsilon < z < 1/\epsilon\}$ . We claim that

$$\lim_{n \rightarrow \infty} \sup_{(u, v, z) \in W_\epsilon} |G_n(u, v, z) - G(u, v, z)| = 0, \quad \forall \epsilon > 0. \quad (5.10)$$

To show (5.10), for given  $\epsilon_1 > 0$  split  $G_n(u, v, z) - G(u, v, z) = \sum_{j=1}^3 \Gamma_{nj}(u, v, z)$ , where, for  $0 < z < n$ ,

$$\begin{aligned}\Gamma_{n1}(u, v, z) &:= \int_{(0,1]^2 \cap D(\epsilon_1)} \{\sqrt{n}g_3([nt] - [nu], [\sqrt{n}s] - [\sqrt{n}v], 1 - \frac{z}{n}) - h_3(t - u, s - v, z)\} dt ds, \\ \Gamma_{n2}(u, v, z) &:= \int_{(0,1]^2 \cap D(\epsilon_1)^c} \sqrt{n}g_3([nt] - [nu], [\sqrt{n}s] - [\sqrt{n}v], 1 - \frac{z}{n}) dt ds, \\ \Gamma_{n3}(u, v, z) &:= - \int_{(0,1]^2 \cap D(\epsilon_1)^c} h_3(t - u, s - v, z) dt ds,\end{aligned}$$

and where the sets  $D(\epsilon), D(\epsilon)^c$  (depending on  $u, v$ ) are defined by

$$D(\epsilon) := \{(t, s) \in (0, 1]^2 : t - u > \epsilon, |s - v| > \epsilon\}, \quad D(\epsilon)^c := (0, 1]^2 \setminus D(\epsilon).$$

To show (5.10), it suffices to verify that for any  $\epsilon > 0, \delta > 0$  there exists  $\epsilon_1 > 0, n_1 \geq 1$  such that

$$\lim_{n \rightarrow \infty} \sup_{(u, v, z) \in W_\epsilon} \Gamma_{n1}(u, v, z) = 0, \quad (5.11)$$

$$\sup_{(u, v, z) \in W_\epsilon} |\Gamma_{ni}(u, v, z)| < \delta, \quad i = 2, 3, \quad \forall n \geq n_1. \quad (5.12)$$

Relation (5.11) follows from Lemma 5.1 Next,  $|\Gamma_{n3}(u, v, z)| \leq C \int_0^{\epsilon_1} t^{-1/2} dt + C \int_{\epsilon_1}^1 t^{-1/2} dt \int_{|s| < \epsilon_1} ds = O(\sqrt{\epsilon_1})$ , implying (5.12) for  $i = 3$  with  $\epsilon_1 = C\delta^2$ . Finally, using (5.5) we obtain  $|\Gamma_{n2}(u, v, z)| \leq C\sqrt{\epsilon_1} + C\sqrt{n} \int_0^1 e^{-c(nt)^{1/3}} dt \leq C\sqrt{\epsilon_1} + C/\sqrt{n} < \delta$  provided  $\sqrt{\epsilon_1} < \delta/(2C), n > n_1 = (2C/\delta)^2$  hold. This proves (5.12) for  $i = 2$  and hence (5.10), too.

Let

$$G'_n(u, v, z) := \sqrt{n} \mathbf{1}(0 < z < n) \int_{(0,1]^2} e^{-z(t-u) - c(n(t-u))^{1/3} - c(\sqrt{n}|s-v|)^{1/2}} \mathbf{1}(t > u) dt ds,$$

where  $c > 0$  is the same as in (5.5). Let us show that

$$J'_n := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G'_n(u, v, z))^\alpha d\mu = o(1). \quad (5.13)$$

Split  $J'_n = \sum_{i=1}^3 I_{ni}$ , where

$$I_{n1} := \int_{(-\infty, 0] \times \mathbb{R}_+ \times \mathbb{R}_+} (G'_n)^\alpha d\mu, \quad I_{n2} := \int_{(0,1] \times [-2,2] \times \mathbb{R}_+} (G'_n)^\alpha d\mu, \quad I_{n3} := \int_{(0,1] \times [-2,2]^c \times \mathbb{R}_+} (G'_n)^\alpha d\mu,$$

$[-2, 2]^c := \mathbb{R} \setminus [-2, 2]$ . Using the fact that  $\int_{\mathbb{R}} e^{-cn^{1/4}|s-v|^{1/2}} dv = C/\sqrt{n}$  and Minkowski's inequality,

$$\begin{aligned}I_{n1} &\leq Cn^{\alpha/2} \left\{ \int_{(0,1]^2} dt ds \left( \int_{\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+} e^{-\alpha z(t+u) - c\alpha(n(t+u))^{1/3} - c\alpha(\sqrt{n}|s-v|)^{1/2}} z^\beta du dv dz \right)^{1/\alpha} \right\}^\alpha \\ &\leq Cn^{\frac{\alpha-1}{2}} \left\{ \int_0^1 dt \left( \int_0^\infty e^{-c\alpha(n(t+u))^{1/3}} \frac{du}{(t+u)^{1+\beta}} \right)^{1/\alpha} \right\}^\alpha \leq Cn^{-(\frac{\alpha+1}{2}-\beta)} I,\end{aligned}$$

where  $\frac{\alpha+1}{2} - \beta > 0$  and  $I := \left\{ \int_0^\infty dt \left( \int_0^\infty e^{-c\alpha(t+u)^{1/3}} (t+u)^{-1-\beta} du \right)^{1/\alpha} \right\}^\alpha < \infty$ . Next,

$$\begin{aligned}I_{n2} &\leq Cn^{\alpha/2} \int_0^\infty z^\beta dz \left\{ \int_{(0,4]^2} e^{-zt - c(nt)^{1/3} - c(\sqrt{n}|s|)^{1/2}} dt ds \right\}^\alpha \\ &\leq C \left\{ \int_0^4 e^{-c(nt)^{1/3}} dt \left( \int_0^\infty e^{-\alpha zt} z^\beta dz \right)^{1/\alpha} \right\}^\alpha \\ &\leq C \left\{ \int_0^\infty e^{-c(nt)^{1/3}} t^{-\frac{1+\beta}{\alpha}} dt \right\}^\alpha \leq Cn^{-(\alpha-1-\beta)} = o(1).\end{aligned}$$

Finally, using  $e^{-c(\sqrt{n}|s-v|)^{1/2}} \leq e^{-(c/2)(\sqrt{n}|v|)^{1/2}}$  for  $|v| \geq 2, |s| \leq 1$  it easily follows  $I_{n3} = O(e^{-c'n^{1/4}}) = o(1) (\exists c' > 0)$ , thus completing the proof of (5.13).

With (5.10) and (5.13) in mind, write

$$\begin{aligned} |J_n - J| &\leq \int_{W_\epsilon} |G_n^\alpha - G^\alpha| d\mu + \int_{W_\epsilon^c} |G_n|^\alpha d\mu + \int_{W_\epsilon^c} |G|^\alpha d\mu \\ &\leq \int_{W_\epsilon} |G_n^\alpha - G^\alpha| d\mu + C \int_{\mathbb{R}^2 \times \mathbb{R}_+} |G'_n|^\alpha d\mu + C \int_{W_\epsilon^c} |\bar{G}|^\alpha d\mu + \int_{W_\epsilon^c} |G|^\alpha d\mu, \end{aligned} \quad (5.14)$$

where  $\bar{G}(u, v, z) := \int_0^1 \int_0^1 \bar{h}_3(t - u, s - v, z) dt ds$ ,  $W_\epsilon^c := \mathbb{R}^2 \times \mathbb{R}_+ \setminus W_\epsilon$ . Since  $G, \bar{G} \in L^\alpha(\mu)$ , the third and fourth terms on the r.h.s. of (5.14) can be made arbitrary small by choosing  $\epsilon > 0$  small enough. Next, for a given  $\epsilon > 0$ , the first term on the r.h.s. of (5.14) vanishes in view of (5.10), and the second term tends to zero, see (5.13). This proves (5.9), thus concluding the proof Theorem 5.1.  $\square$

The next Theorem 5.2 shows that when partial sums of  $\{\mathfrak{X}_3(t, s)\}$  in (5.1) are taken on ‘commensurate’ rectangles (the number of summands in the horizontal and the vertical directions grow at the same rate  $O(n)$ ) the limit field is different.

**Theorem 5.2** *Assume the conditions and notation of Theorem 5.1. Then*

$$n^{-H_*} \sum_{t=1}^{[nx]} \sum_{s=1}^{[ny]} \mathfrak{X}_3(t, s) \xrightarrow{\text{fdd}} V_{3*}(x, y), \quad x, y > 0, \quad H_* := \frac{1 + \alpha - \beta}{\alpha} \quad (5.15)$$

where

$$V_{3*}(x, y) := \int_{\mathbb{R}^2 \times \mathbb{R}_+} \mathcal{M}(du, dv, dz) \mathbf{1}(0 < v \leq y) \int_0^x h_{3*}(t - u, z) dt, \quad (5.16)$$

$$h_{3*}(u, z) := \int_{\mathbb{R}} h_3(u, v, z) dv = 12e^{-3uz} \mathbf{1}(u > 0), \quad (5.17)$$

where  $\mathcal{M}$  is the same as in Theorem 5.1.

*Proof.* Similarly as in the case of Theorem 5.1, we prove one-dimensional convergence in (5.15) at  $x = y = 1$  only, and assume  $\Phi(da) = (1 - a)^\beta da$ . Correspondingly, it suffices to show the limit  $\lim J_{n*} = J_*$ , where

$$J_{n*} := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_{n*}(u, v, z))^\alpha d\mu(u, v, z), \quad J_* := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_*(u, v, z))^\alpha d\mu(u, v, z),$$

where

$$\begin{aligned} G_*(u, v, z) &:= \mathbf{1}(0 < v < 1) \int_0^1 dt \int_{\mathbb{R}} ds h_3(t - u, s, z), \\ G_{n*}(u, v, z) &:= \int_0^1 dt \sum_{s=1}^n g_3([nt] - [nu], s - [nv], 1 - \frac{z}{n}) \mathbf{1}(0 < z < n), \\ &= \int_0^1 dt \int_{\mathbb{R}} ds \sqrt{n} g_3([nt] - [nu], [\sqrt{n}s], 1 - \frac{z}{n}) \mathbf{1}(0 < z < n, 1 - [nv] \leq [\sqrt{n}s] \leq n - [nv]), \\ &=: \int_0^1 dt \int_{\mathbb{R}} ds f_n(t, s, u, v, z). \end{aligned}$$

Define  $J'_{n*} := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_{n*}(u, v, z))^\alpha \mathbf{1}(|v| \leq 3) d\mu$ ,  $J''_{n*} := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_{n*}(u, v, z))^\alpha \mathbf{1}(|v| > 3) d\mu$ ,  $J'_{n*} + J''_{n*} = J_{n*}$ . Then  $\lim J_{n*} = J_*$  follows from  $\lim J'_{n*} = J_*$  and  $\lim J''_{n*} = 0$ .

Note that for any  $u \in \mathbb{R}$ ,  $u < t$ ,  $v \in \mathbb{R} \setminus \{0, 1\}$ ,  $s, z > 0$ , we have pointwise convergence

$$\begin{aligned} \mathbf{1}(1 - [nv] \leq [\sqrt{n}s] \leq n - [nv]) &\rightarrow \mathbf{1}(0 < v < 1), \text{ as } n \rightarrow \infty, \\ \sqrt{n} g_3([nt] - [nu], [\sqrt{n}s], 1 - \frac{z}{n}) &\rightarrow h_3(t - u, s, z), \text{ as } n \rightarrow \infty, \end{aligned}$$

and therefore

$$f_n(t, s, u, v, z) \rightarrow h_3(t - u, s, z) \mathbf{1}(0 < v < 1), \text{ as } n \rightarrow \infty. \quad (5.18)$$

We claim that for any  $u \in \mathbb{R}$ ,  $v \in \mathbb{R} \setminus \{0, 1\}$ ,  $z > 0$ ,

$$G_{n*}(u, v, z) \rightarrow G_*(u, v, z), \quad \text{as } n \rightarrow \infty. \quad (5.19)$$

To show (5.19), for given  $\epsilon_1 > 0$  split  $G_{n*}(u, v, z) - G_*(u, v, z) = \sum_{j=1}^3 \Gamma_{nj}^*(u, v, z)$ , where, for  $0 < z < n$ ,

$$\begin{aligned} \Gamma_{n1}^*(u, v, z) &:= \int_0^1 \int_{|s| > \epsilon_1} (f_n(t, s, u, v, z) - h_3(t - u, s, z) \mathbf{1}(0 < v < 1)) dt ds, \\ \Gamma_{n2}^*(u, v, z) &:= \int_0^1 \int_{|s| \leq \epsilon_1} f_n(t, s, u, v, z) dt ds, \\ \Gamma_{n3}^*(u, v, z) &:= - \int_0^1 \int_{|s| \leq \epsilon_1} h_3(t - u, s, z) \mathbf{1}(0 < v < 1) dt ds, \end{aligned}$$

To show (5.19), it suffices to verify that for any  $\epsilon > 0$ ,  $\delta > 0$  there exists  $\epsilon_1 > 0$ ,  $n_1 \geq 1$  such that

$$\lim_{n \rightarrow \infty} \Gamma_{n1}^*(u, v, z) = 0, \quad (5.20)$$

$$|\Gamma_{ni}^*(u, v, z)| < \delta, \quad i = 2, 3, \quad \forall n \geq n_1. \quad (5.21)$$

Relation (5.21) follows from Lemma 5.1,  $|\Gamma_{n2}^*(u, v, z)| \leq C_u \epsilon_1 + C_u \epsilon_1 \sqrt{n} \int_0^1 e^{-c(nt)^{1/3}} dt \leq C_u \epsilon_1 + C_u \epsilon_1 / \sqrt{n} < \delta$  provided  $\epsilon_1 < \delta / (2C_u)$ .  $|\Gamma_{n3}^*(u, v, z)| \leq C_u \epsilon_1$ , implying (5.21) for  $i = 3$  with  $\epsilon_1 = \delta / C_u$ . Relation (5.20) follows from (5.18) and the dominated convergence theorem. For this we need to find the dominated integrable function for  $f_n(t, s, u, v, z)$ . Using inequality from Lemma 5.1 and inequalities  $e^{-x} \leq x^{-3/2}$ , for  $x > 0$ , and  $\sqrt{x}e^{-x} \leq e^{-x/2}$ , for  $x > 0$ , we have for fixed  $u$ ,  $(t - u > 0)$ ,  $v$ ,  $z$ :

$$\begin{aligned} |f_n(t, s, u, v, z)| &\leq C \frac{1}{\sqrt{\frac{[nt] - [nu]}{n}}} e^{-\frac{s^2}{16 \frac{[nt] - [nu]}{n}}} + C \sqrt{n} e^{-cn^{1/3}(t-u)^{1/3} - c|s|^{1/2}} \\ &\leq C \frac{1}{|s|} e^{-\frac{s^2}{24 \frac{[nt] - [nu]}{n}}} + C \sqrt{n} (n^{1/3}(t-u)^{1/3})^{-3/2} e^{-c|s|^{1/2}} \\ &\leq \frac{1}{|s|} e^{-\frac{s^2}{24(1+|u|)}} + C(t-u)^{-1/2} e^{-c|s|^{1/2}} =: \bar{f}(t, s). \end{aligned}$$

It is not difficult to see, that  $\int_0^1 \int_{|s| > \epsilon_1} \bar{f}(t, s) dt ds < \infty$ . Therefore pointwise convergence in (5.19) is proved. Using (5.5), we also get

$$\begin{aligned} G_{n*}(u, v, z) &= \int_0^1 dt \int_{\mathbb{R}} ds f_n(t, s, u, v, z) \\ &\leq \int_0^1 dt \int_{\mathbb{R}} ds (\bar{h}_3\left(\frac{[nt] - [nu]}{n}, s, z\right) + \sqrt{n} e^{-z \frac{[nt] - [nu]}{n} - c([nt] - [nu])^{1/3} - c(\sqrt{n}|s|)^{1/2}}) \\ &\leq C \int_0^1 dt e^{-z(t-u)} \mathbf{1}(u < t) \end{aligned}$$

The integral of the function on the right side of last inequality is finite. Indeed,

$$\int_{\mathbb{R}^2 \times \mathbb{R}_+} \left( \int_0^1 dt e^{-z(t-u)} \right)^\alpha \mathbf{1}(u < t, |v| \leq 3) z^\beta du dv dz \leq C \int_{\mathbb{R}} du \int_0^\infty dz z^\beta \left( \int_0^1 dt e^{-z(t-u)} \right)^\alpha \mathbf{1}(u < t) =: I_1 + I_2.$$

$$\begin{aligned} I_1 &\leq C \int_0^1 du \int_0^\infty dz z^\beta \left( \int_u^1 dt e^{-z(t-u)} \right)^\alpha \leq C \int_0^1 du \int_0^\infty dz z^{\beta-\alpha} (1 - e^{-z(1-u)})^\alpha \\ &\leq C \int_0^\infty z^{\beta-\alpha} (1 - e^{-z})^\alpha dz \leq C \\ I_2 &\leq C \int_0^{+\infty} du \int_0^\infty dz z^\beta \left( \int_0^1 dt e^{-z(t+u)} \right)^\alpha \leq C \left\{ \int_0^1 dt \left( \int_0^{+\infty} du \int_0^\infty dz z^\beta e^{-\alpha z(t+u)} \right)^{1/\alpha} \right\}^\alpha \\ &\leq C \left\{ \int_0^1 dt \left( \int_0^{+\infty} (u+t)^{-1-\beta} du \right)^{1/\alpha} \right\}^\alpha \leq C \left\{ \int_0^1 t^{-\beta/\alpha} dt \right\}^\alpha \leq C, \text{ since } 1 - \beta/\alpha > 0. \end{aligned}$$

From the last fact, the limit in (5.19) and the dominated convergence theorem follows  $\lim J'_{n\star} = J_\star$ . Now we will show  $\lim J''_{n\star} = 0$ . Again using inequality in (5.5), we have  $J''_{n\star} \leq I_{1,n} + I_{2,n}$ , where

$$\begin{aligned} I_{1,n} &:= \int_{\mathbb{R}} du \int_{|v|>3} dv \int_0^\infty dz z^\beta \left( \int_0^1 dt \int_{\mathbb{R}} ds \sqrt{n} e^{-z \frac{[nt]-[nu]}{n} - c([nt]-[nu])^{1/3} - c(\sqrt{n}|s|)^{1/2}} \right)^\alpha \mathbf{1}_n(t, u, z, s, v), \\ I_{2,n} &:= \int_{\mathbb{R}} du \int_{|v|>3} dv \int_0^\infty dz z^\beta \left( \int_0^1 dt \int_{\mathbb{R}} ds \bar{h}_3 \left( \frac{[nt] - [nu]}{n}, s, z \right) \right)^\alpha \mathbf{1}_n(t, u, z, s, v), \end{aligned}$$

here  $\mathbf{1}_n(t, u, z, s, v) := \mathbf{1}([nt] - [nu] > 0, 0 < z < n, 1 - [nv] \leq [\sqrt{n}s] \leq n - [nv])$ . Note that

$$\int_{\mathbb{R}} ds e^{-c(\sqrt{n}|s|)^{1/2}} \mathbf{1}(1 - [nv] \leq [\sqrt{n}s] \leq n - [nv], |v| > 3) \leq C\sqrt{n} e^{-c\sqrt{n}(\min(|v|, |v-2|))^{1/2}} \mathbf{1}(|v| > 3).$$

Therefore,

$$\begin{aligned} I_{1,n} &\leq Cn^\alpha \int_{|v|>3} e^{-c\alpha\sqrt{n}(\min(|v|, |v-2|))^{1/2}} dv \int_{\mathbb{R}} du \int_0^\infty dz z^\beta \left( \int_0^1 dt e^{-z(t-u) - c(t-u)^{1/3}} \right)^\alpha \mathbf{1}(t-u > 0, 0 < z < n) \\ &\leq Cn^{\alpha+\beta+1} \int_{|v|>1} e^{-c\alpha\sqrt{n}|v|^{1/2}} dv \int_{\mathbb{R}} du \left( \int_0^1 dt e^{-c(t-u)^{1/3}} \right)^\alpha \mathbf{1}(t-u > 0) \\ &\leq Cn^{\alpha+\beta} e^{-c\sqrt{n}} \rightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned}$$

$$\begin{aligned} I_{2,n} &\leq Cn^{\frac{\alpha}{2}} \int_{\mathbb{R}} du \int_{|v|>1} dv \int_0^\infty dz z^\beta \left( \int_0^1 dt \frac{1}{\sqrt{t-u}} e^{-z(t-u) - c\frac{nv^2}{t-u}} \right)^\alpha \mathbf{1}(t-u > 0, 0 < z < n) \\ &\leq Cn^{\frac{\alpha}{2}} \left( \int_0^1 dt \left( \int_{\mathbb{R}} du \int_{|v|>1} dv \int_0^\infty dz z^\beta (t-u)^{-\frac{\alpha}{2}} e^{-z\alpha(t-u) - c\alpha\frac{nv^2}{t-u}} \right)^{\frac{1}{\alpha}} \right)^\alpha \mathbf{1}(t-u > 0, 0 < z < n) \\ &\leq Cn^{\frac{\alpha}{2}} \left( \int_0^1 dt \left( \int_{\mathbb{R}} du \int_{|v|>1} dv (t-u)^{-\frac{\alpha}{2}-\beta-1} e^{-c\alpha\frac{nv^2}{t-u}} \right)^{\frac{1}{\alpha}} \right)^\alpha \mathbf{1}(t-u > 0) \\ &\leq Cn^{-\beta} \int_{v>1} dv v^{-2(\frac{\alpha}{2}+\beta)} \int_0^\infty dy y^{-\frac{\alpha}{2}-\beta-1} e^{-\frac{c\alpha}{y}} = Cn^{-\beta} \rightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned}$$

since  $1 - 2(\frac{\alpha}{2} + \beta) < 0$  and  $\int_0^\infty y^{-\frac{\alpha}{2}-\beta-1} e^{-\frac{c\alpha}{y}} dy < \infty$ . This proves  $\lim J''_{n\star} = 0$  and Theorem 5.2 too.

**Remark 5.2** It is not difficult to show that the random fields  $\{V_3(x, y)\}$  and  $\{V_{3\star}(x, y)\}$  in Theorems 5.1 and 5.2 are related by  $\lambda^{-1/\alpha} V_3(x, \lambda y) \xrightarrow{\text{fdd}} V_{3\star}(x, y)$ ,  $x, y > 0$ ,  $\lambda \rightarrow \infty$ .

**Proposition 5.1** *Let the conditions of Theorem 5.1 be satisfied. Then:*

- (i) *The random field  $\{\mathfrak{X}_3(t, s)\}$  in (5.1) has anisotropic distributional long memory with parameters  $H_1 = H = \frac{\frac{1}{2} + \alpha - \beta}{\alpha}$ ,  $H_2 = 2H_1$ .*
- (ii) *The random field  $\{\mathfrak{X}_3(t, s)\}$  in (5.1) does not have isotropic distributional long memory.*

*Proof.* (i) With Theorem 5.1 in mind, it suffices to check that the random field  $\{V_3(x, y)\}$  in (5.2) has dependent increments in arbitrary direction. To this end, consider arbitrary rectangles  $K_i = K_{(\xi_i, \eta_i); (x_i, y_i)} \subset \mathbb{R}_+^2$ ,  $i = 1, 2$ , and write  $\int = \int_{\mathbb{R}^2 \times \mathbb{R}_+}$ . Then  $V_3(K_i) = \int G_{K_i}(u, v, z) d\mathcal{M}$ , where  $G_{K_i}(u, v, z) := \int_{K_i} h_3(t-u, s-v, z) dt ds$ . Note  $G_{K_i} \geq 0$  and  $G_{K_i}(u, v, z) > 0$  for any  $u < x_i$  implying  $\text{supp}(G_{K_1}) \cap \text{supp}(G_{K_2}) \neq \emptyset$ . Hence and from ([39], Th 3.5.3, p. 128) it follows that the increments  $V_3(K_i)$ ,  $i = 1, 2$  on arbitrary nonempty rectangles  $K_1, K_2$  are dependent, thus concluding the proof of (i).

(ii) With Theorem 5.2 in mind, it suffices to check that the random field  $\{V_{3\star}(x, y)\}$  in (5.16) has independent increments in the vertical directions. Similarly as in the proof of (i), for any rectangle  $K = K_{(\xi, \eta); (x, y)} \subset \mathbb{R}_+^2$ ,  $V_{3\star}(K) = \int G_K^*(u, v, z) d\mathcal{M}$ , where  $G_K^*(u, v, z) := \mathbf{1}(\eta < v \leq y) \int_\xi^\eta h_{3\star}(t-u, z) dt$ . Clearly, if  $K_i$ ,  $i = 1, 2$  are two rectangle separated by a horizontal line, then  $\text{supp}(G_{K_1}) \cap \text{supp}(G_{K_2}) = \emptyset$ , implying the independence of  $V_{3\star}(K_1)$  and  $V_{3\star}(K_2)$ . Proposition 5.1 is proved.  $\square$



Let  $\alpha = 2$  and  $r_3(t, s) = \mathbb{E}\mathfrak{X}_3(t, s)\mathfrak{X}_3(0, 0)$  be the covariance function of the aggregated Gaussian random field in (5.1). Using the representation of  $r_3(t, s)$  in (1.11) and Lemma 5.1, the following proposition obtains the asymptotics of  $r_3(t, s)$  as  $|t| + |s| \rightarrow \infty$ .

**Proposition 5.2** *Assume  $\alpha = 2$  and the conditions of Theorem 5.1. Then for any  $(t, s) \in \mathbb{R}_0^2$*

$$\lim_{\lambda \rightarrow \infty} \lambda^{\beta+1/2} r_3([\lambda t], [\sqrt{\lambda} s]) = \rho(t, s) := \begin{cases} C_3 |s|^{-2\beta-1} \gamma(\beta + 1/2, s^2/4|t|), & t \neq 0, s \neq 0, \\ C_3 |s|^{-2\beta-1} \Gamma(\beta + 1/2), & t = 0, \\ C_4 |t|^{-\beta-1/2}, & s = 0 \end{cases} \quad (5.22)$$

and

$$\lim_{\lambda \rightarrow \infty} \lambda^{\beta+1/2} r_3([\lambda t], [\lambda s]) = \rho_*(t, s) := \begin{cases} 0, & s \neq 0, \\ C_4 |t|^{-\beta-1/2}, & s = 0, t \neq 0, \end{cases} \quad (5.23)$$

where  $\gamma(\alpha, x) := \int_0^x y^{\alpha-1} e^{-y} dy$  is incomplete gamma function and  $C_3 = \pi^{-\frac{1}{2}} 2^{2\beta-1} 3^{1-\beta} \sigma^2 \phi_1 \Gamma(\beta + 1)$ ,  $C_4 = 4^{-\frac{1}{2}-\beta} (\frac{1}{2} + \beta)^{-1} C_3$ .

Notice that under the ‘parabolic scaling’ in (5.22) we have a non-degenerated limit  $\rho(t, s)$  which is a generalized homogeneous function (see, e.g., [22] for a general account) satisfying  $\lambda^{2(1+\frac{H_1}{H_2}-H_1)} \rho(\lambda t, \lambda^{H_1/H_2} s) = \rho(t, s) \forall \lambda > 0$  with  $H_1, H_2$  as in Proposition 5.1 (i) ( $\alpha = 2$ ). On the other hand, the ‘isotonic scaling’ in (5.23) leads to a degenerated limit concentrated on the anisotropy axis  $s = 0$  of the 3N model and vanishing elsewhere. It is clear that the corresponding integrated Gaussian random field must have independent increments in the vertical direction, in accordance with Proposition 5.1 (ii).

*Proof of Proposition 5.2.* We have

$$r_3(t, s) = \sigma^2 \sum_{(u,v) \in \mathbb{Z}^2} \int_{[0,1]} g_3(t+u, s+v, a) g_3(u, v, a) \Phi(da), \quad (t, s) \in \mathbb{Z}^2, \quad (5.24)$$

where  $\sigma^2 = \mathbb{E}\varepsilon^2$ . For ease of notation, assume  $\phi(a) = (1-a)^\beta, a \in [0, 1]$  in the rest of the proof. Then

$$\begin{aligned} r_3([\lambda t], [\sqrt{\lambda} s]) &= \sigma^2 \int_0^\infty du \int_{\mathbb{R}} dv \int_0^1 (1-a)^\beta da g_3([u], [v], a) g_3([\lambda t] + [u], [\sqrt{\lambda} s] + [v], a) \\ &= \lambda^{1/2-\beta} \sigma^2 \int_0^\infty dx \int_{\mathbb{R}} dy \int_0^\lambda z^\beta dz g_3([\lambda x], [\sqrt{\lambda} y], 1 - \frac{z}{\lambda}) g_3([\lambda t] + [\lambda x], [\sqrt{\lambda} s] + [\sqrt{\lambda} y], 1 - \frac{z}{\lambda}). \end{aligned}$$

Hence,

$$\lambda^{\beta+1/2} r_3([\lambda t], [\sqrt{\lambda} s]) = \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \mathcal{K}_\lambda(x, y, z) d\mu,$$

where  $d\mu(x, y, z) = z^\beta dx dy dz$  and

$$\mathcal{K}_\lambda(x, y, z) := \lambda \sigma^2 g_3([\lambda x], [\sqrt{\lambda} y], 1 - \frac{z}{\lambda}) g_3([\lambda t] + [\lambda x], [\sqrt{\lambda} s] + [\sqrt{\lambda} y], 1 - \frac{z}{\lambda}) \mathbf{1}(0 < z < \lambda).$$

By Lemma 5.1, for any  $(x, y, z) \in (0, \infty) \times \mathbb{R} \times (0, \infty)$  fixed,  $\mathcal{K}_\lambda(x, y, z) \rightarrow \mathcal{K}(x, y, z) := \sigma^2 h_3(x, y, z) h_3(x+t, y+s, z)$ ,

where

$$\begin{aligned}
\int_{\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+} \mathcal{K}(x, y, z) d\mu &= \sigma^2 \int_0^\infty dx \int_{\mathbb{R}} dy \int_0^\infty z^\beta dz \frac{3}{2\sqrt{\pi x}} e^{-3zx - \frac{y^2}{4x}} \frac{3}{2\sqrt{\pi(t+x)}} e^{-3z(t+x) - \frac{(s+y)^2}{4(t+x)}} \\
&= \frac{9\sigma^2}{4\pi} \int_0^\infty dx \left\{ \int_0^\infty z^\beta e^{-3z(2x+t)} dz \right\} \left\{ \int_{\mathbb{R}} \frac{1}{\sqrt{x(t+x)}} e^{-\frac{y^2}{4x}} e^{-\frac{(s-y)^2}{4(t+x)}} dy \right\} \\
&= \frac{9\sigma^2}{4\pi} \int_0^\infty dx \left\{ \frac{\Gamma(\beta+1)}{(3(2x+t))^{1+\beta}} \right\} \left\{ \frac{2\sqrt{\pi}}{\sqrt{2x+t}} e^{-\frac{s^2}{4(2x+t)}} \right\} \\
&= \frac{9\sigma^2 \Gamma(\beta+1)}{2\sqrt{\pi} 3^{1+\beta}} \int_0^\infty \frac{1}{(2x+t)^{3/2+\beta}} e^{-\frac{s^2}{4(2x+t)}} dx \\
&= \frac{3^{1-\beta} \sigma^2 \Gamma(\beta+1)}{4\sqrt{\pi}} \int_t^\infty \frac{1}{x^{3/2+\beta}} e^{-\frac{s^2}{4x}} dx \\
&= \begin{cases} \frac{3^{1-\beta} \sigma^2 \Gamma(\beta+1)}{4^{1/2-\beta} \sqrt{\pi}} |s|^{-2\beta-1} \gamma(\beta+1/2, s^2/4t), & s \neq 0, \\ \frac{3^{1-\beta} \sigma^2 \Gamma(\beta+1)}{4\sqrt{\pi}(\frac{1}{2}+\beta)} t^{-\beta-1/2}, & s = 0. \end{cases}
\end{aligned}$$

The legitimacy of the passage to the limit  $\lambda \rightarrow \infty$  under the sign of the integral follows from Lemma 5.1. Indeed, the bound (5.5) implies  $|\mathcal{K}_\lambda(x, y, z)| \leq C(\mathcal{K}'(x, y, z) + \mathcal{K}_\lambda''(x, y, z))$ , where  $0 \leq \mathcal{K}'(x, y, z) := \bar{h}_3(x, y, z) \bar{h}_3(x+t, y+s, z)$  does not depend on  $\lambda$  and satisfies  $\int \mathcal{K}'(x, y, z) d\mu < \infty$ , see above, while

$$0 \leq \mathcal{K}_\lambda''(x, y, z) := \lambda e^{-zx - c(\lambda x)^{1/3} - c(\sqrt{\lambda}|y|)^{1/2}} e^{-z(x+t) - c(\lambda(x+t))^{1/3} - c(\sqrt{\lambda}|s+y|)^{1/2}}$$

satisfies  $\lim_{\lambda \rightarrow \infty} \int \mathcal{K}_\lambda''(x, y, z) d\mu = 0$  for any  $(t, s) \in \mathbb{R}_0^2$  fixed. The last fact can be easily verified by separately considering the two cases  $t > 0$  and  $t = 0, s \neq 0$ . E.g., in the first case, we have  $\mathcal{K}_\lambda''(x, y, z) \leq \lambda e^{-c(\lambda t)^{1/3}} e^{-zx - c(\lambda x)^{1/3} - c(\sqrt{\lambda}|y|)^{1/2}}$  and  $\int \mathcal{K}_\lambda''(x, y, z) d\mu \leq C e^{-c'(\lambda t)^{1/3}}, 0 < c' < c$  easily follows. The convergence in (5.23) can be proved in a similar way. Proposition 5.2 is proved.

## 6 Aggregation of the 4N model

The stationary solution of (1.16) is given by

$$X_4(t, s) = \sum_{(u, v) \in \mathbb{Z}^2} g_4(t - u, s - v, A) \varepsilon(u, v), \quad (t, s) \in \mathbb{Z}^2, \quad (6.1)$$

where

$$g_4(t, s, a) = \sum_{k=0}^\infty a^k p_k(t, s), \quad p_k(t, s) = P(W_k = (t, s) | W_0 = (0, 0)) \quad (6.2)$$

and  $\{W_k\}$  is a random walk on  $\mathbb{Z}^2$  with one-step transition probabilities in Fig. 1 c). Under the assumptions of Proposition 3.2, the aggregated random field of (6.1) exists and is written as

$$\mathfrak{X}_4(t, s) = \sum_{(u, v) \in \mathbb{Z}^2} \int_0^1 g_4(t - u, s - v, a) M_{u, v}(da), \quad (t, s) \in \mathbb{Z}^2, \quad (6.3)$$

where  $\{M_{u, v}(da), (u, v) \in \mathbb{Z}^2\}$  is the same  $\alpha$ -stable random measure as in Sec. 3. For  $1 < \alpha \leq 2$  and a regularly varying mixing density as in (1.13), the random field in (6.3) is well-defined under the same condition  $0 < \beta < \alpha - 1$  as in Theorem 5.1. Recall  $\mathbb{R}_0^2 = \mathbb{R}^2 \setminus \{(0, 0)\}$ .

**Lemma 6.1** *For any  $(t, s, z) \in \mathbb{R}_0^2 \times (0, \infty)$*

$$\lim_{\lambda \rightarrow \infty} g_4([\lambda t], [\lambda s], 1 - \frac{z}{\lambda^2}) = h_4(t, s, z) = \frac{2}{\pi} K_0(2\sqrt{z(t^2 + s^2)}). \quad (6.4)$$

*The convergence in (6.4) is uniform on any relatively compact set  $\{\epsilon < |t| + |s| < 1/\epsilon\} \times \{\epsilon < z < 1/\epsilon\} \subset \mathbb{R}_0^2 \times \mathbb{R}_+, \epsilon > 0$ .*

Moreover, there exists constants  $C, c > 0$  such that for all sufficiently large  $\lambda$  and any  $(t, s, z) \in \mathbb{R}_0^2 \times (0, \lambda^2)$  the following inequality holds:

$$g_4([\lambda t], [\lambda s], 1 - \frac{z}{\lambda^2}) < C \left\{ h_4(t, s, z) + e^{-c\sqrt{\lambda}(|t|^{1/2} + |s|^{1/2})} \right\}. \quad (6.5)$$

The main result of this sec. is Theorem 6.1 below.

**Theorem 6.1** *Let  $\{\varepsilon(t, s)\}$  and  $\Phi$  satisfy the same conditions as in Theorem 5.1, and  $\{\mathfrak{X}_4(t, s)\}$  be the aggregated  $4N$  random field in (6.3). Then*

$$n^{-H} \sum_{t=1}^{[nx]} \sum_{s=1}^{[ny]} \mathfrak{X}_4(t, s) \xrightarrow{\text{fdd}} V_4(x, y), \quad x, y > 0, \quad (6.6)$$

where  $H := \frac{2(\alpha-\beta)}{\alpha}$  and

$$V_4(x, y) := \int_{\mathbb{R}^2 \times \mathbb{R}_+} \mathcal{M}(du, dv, dz) \int_0^x \int_0^y h_4(t-u, s-v, z) dt ds \quad (6.7)$$

and where  $\mathcal{M}$  is the same  $\alpha$ -stable random measure on  $\mathbb{R}^2 \times \mathbb{R}_+$  as in Theorem 5.1 and  $h_4(t, s, z)$  is given in (6.4).

*Proof.* As in all previous theorems, we prove the convergence of one-dimensional distributions in (6.6) at  $x = y = 1$ . Accordingly, it suffices to show the limit  $\lim J_n = J$ , where

$$J_n := \frac{1}{n^{H\alpha}} \sum_{(u,v) \in \mathbb{Z}^2} \mathbb{E} \left( \sum_{t,s=1}^n g_4(t-u, s-v, A) \right)^\alpha, \quad J := \int_{\mathbb{R}^2 \times \mathbb{R}_+} \left( \int_{(0,1]^2} h_4(t-u, s-v, z) dt ds \right)^\alpha d\mu.$$

Let us first check that  $J = C \int_{\mathbb{R}^2 \times \mathbb{R}_+} \left( \int_{(0,1]^2} K_0(2\sqrt{z}\|v-w\|) dv \right)^\alpha z^\beta dw dz < \infty$ . Here,  $\|x\|^2 := x_1^2 + x_2^2$ , for  $x = (x_1, x_2) \in \mathbb{R}^2$ . To this end, split  $J = J_1 + J_2$ , where  $J_1 := \int_{\{\|w\| \leq \sqrt{2}\} \times \mathbb{R}_+} \cdots$ ,  $J_2 := \int_{\{\|w\| > \sqrt{2}\} \times \mathbb{R}_+} \cdots$ . By Minkowski inequality,

$$\begin{aligned} J_2 &\leq C \left\{ \int_{\{\|v\| \leq \sqrt{2}\}} dv \left[ \int_{\{\|w\| > \sqrt{2}\} \times \mathbb{R}_+} K_0^\alpha(2\sqrt{z}\|v-w\|) z^\beta dz dw \right]^{1/\alpha} \right\}^\alpha \\ &\leq C \left\{ \int_{\{\|v\| \leq \sqrt{2}\}} dv \left[ \int_{\{\|w\| > \sqrt{2}\}} \|v-w\|^{-2-2\beta} dw \right]^{1/\alpha} \right\}^\alpha \\ &\leq C \left\{ \int_{\{\|v\| \leq \sqrt{2}\}} (\sqrt{2} - \|v\|)^{-2\beta/\alpha} dv \right\}^\alpha < \infty, \end{aligned}$$

where we used the facts that  $\int_0^\infty K_0^\alpha(2\sqrt{z}) z^\beta dz < \infty$  and  $0 < \beta < \alpha - 1 \leq 2$ . Next,

$$\begin{aligned} J_1 &\leq C \int_{\{\|w\| \leq \sqrt{2}\}} dw \int_0^\infty z^\beta dz \left( \int_{\{\|v\| \leq \sqrt{2}\}} K_0(2\sqrt{z}\|v\|) dv \right)^\alpha \\ &\leq C \int_0^\infty z^\beta dz \left( \int_0^{\sqrt{2}} K_0(2\sqrt{z}r) r dr \right)^\alpha \\ &\leq C \int_0^\infty z^\beta (z^{-\alpha/2} \mathbf{1}(0 < z < 1) + z^{-\alpha} \mathbf{1}(z \geq 1)) dz < \infty, \end{aligned}$$

where we used  $0 < \beta < \alpha - 1$  and the inequality

$$\int_0^{\sqrt{2}} K_0(2\sqrt{z}r) r dr \leq C \begin{cases} z^{-1/2}, & 0 < z \leq 1, \\ z^{-1}, & z > 1. \end{cases}$$

which is a consequence of the fact that the function  $r \mapsto rK_0(r)$  is bounded and integrable on  $(0, \infty)$ . This proves  $J < \infty$ .

Next, we prove the convergence  $J_n \rightarrow J$ . The proof uses Lemma 6.1. Assume for simplicity  $\phi(a) = (1-a)^\beta$ . Then

$$J_n = \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_n(u, v, z))^\alpha d\mu(u, v, z), \quad J = \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G(u, v, z))^\alpha d\mu(u, v, z),$$

where

$$G(u, v, z) := \int_{(0,1]^2} h_4(t-u, s-v, z) dt ds, \quad G_n(u, v, z) := \int_{(0,1]^2} g_4([nt] - [nu], [ns] - [nv], 1 - \frac{z}{n^2}) dt ds.$$

Let  $G'_n(u, v, z) := \mathbf{1}(0 < z < n^2) \int_{(0,1]^2} e^{-c(\sqrt{n|t-u|} + \sqrt{n|s-v|})} dt ds$ , where  $c > 0$  is the same as in (6.5). Then

$$J'_n := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G'_n(u, v, z))^\alpha d\mu(u, v, z) = O(n^{2(\beta-\alpha+1)}) = o(1). \quad (6.8)$$

Indeed,  $J'_n \leq Cn^{2\beta+2} \{ \int_{\mathbb{R}} (\int_0^1 e^{-c\sqrt{n|t-u|}} dt)^\alpha du \}^2$ , where  $\int_{\mathbb{R}} (\int_0^1 e^{-c\sqrt{n|t-u|}} dt)^\alpha du \leq \int_{\{|u|<2\}} (\dots)^\alpha du + \int_{\{|u|\geq 2\}} (\dots)^\alpha du =: i'_n + i''_n$ . Here,  $i'_n \leq C(\int_0^3 e^{-c\sqrt{nv}} dv)^\alpha \leq C/n^\alpha$  and  $i''_n \leq C \int_2^\infty e^{-c\alpha\sqrt{n(u-1)}} du = O(e^{-c'\sqrt{n}})$ ,  $c' > 0$ . This proves (6.8). The rest of the proof is similar as in the case of Theorem 5.1. Theorem 6.1 is proved.  $\square$

**Proposition 6.1** *Let the conditions of Theorem 6.1 be satisfied. Then the random field  $\{\mathfrak{X}_4(t, s)\}$  in (6.3) has isotropic distributional long memory.*

*Proof.* Similar to the proof of Proposition 5.1 we need to show that the random field  $\{V_4(x, y)\}$  in (6.7) has dependent increments in arbitrary direction. Consider arbitrary rectangles  $K_i = K_{(\xi_i, \eta_i); (x_i, y_i)} \subset \mathbb{R}_+^2$ ,  $i = 1, 2$ , and write  $\int = \int_{\mathbb{R}^2 \times \mathbb{R}_+}$ . Then  $V_4(K_i) = \int G_{K_i}(u, v, z) d\mathcal{M}$ , where

$$G_{K_i}(u, v, z) := \int_{K_i} h_4(t-u, s-v, z) dt ds = \int_{K_i} \left( \frac{1}{\pi} \int_0^\infty \frac{1}{x} \exp\left\{-zx - \frac{(t-u)^2 + (s-v)^2}{x}\right\} dx \right) dt ds > 0, \quad i = 1, 2.$$

Therefore  $\text{supp}(G_{K_1}) \cap \text{supp}(G_{K_2}) \neq \emptyset$ . Hence it follows that the increments  $V_4(K_i)$ ,  $i = 1, 2$  on arbitrary nonempty rectangles  $K_1, K_2$  are dependent and random field in (6.7) has isotropic long memory.  $\square$

The following proposition obtains an asymptotic behavior of the covariance function of the Gaussian aggregated random field in (6.7) ( $\alpha = 2$ ). The proof of Proposition 6.2 uses Lemma 6.1 and is omitted.

**Proposition 6.2** *Assume  $\alpha = 2$  and the conditions of Theorem 6.1. Then for any  $(t, s) \in \mathbb{R}_0^2$*

$$\lim_{\lambda \rightarrow \infty} \lambda^{2\beta} r_4([\lambda t], [\lambda s]) = \frac{\sigma^2 \phi_1 \Gamma(\beta+1) \Gamma(\beta)}{\pi} (t^2 + s^2)^{-\beta}. \quad (6.9)$$

## 7 Appendix. Proofs of Lemmas 4.1, 5.1 and 6.1

Let us note that the asymptotics of some lattice Green functions as  $|t| + |s| \rightarrow \infty$  and  $a \nearrow 1$  simultaneously was derived in Montroll and Weiss [33] using Laplace's method, see, e.g., ([33], (II.16)), ([24], (3.185)), however in the literature we did not find dominating bounds needed for our purposes. As noted in Sec. 1, our proofs use probabilistic tools and are completely independent.

*Proof of Lemma 4.1.* In terms of  $\tilde{g}_2$  of (4.5), the statements of the Lemma 4.1 write as

$$\lim_{\lambda \rightarrow \infty} \sup_{\substack{\epsilon < u, z, |v| < 1/\epsilon \\ ([\lambda u], [\sqrt{\lambda} v]) \in \mathbb{Z}^2}} |\sqrt{\lambda} \tilde{g}_2([\lambda u], [\sqrt{\lambda} v], 1 - \frac{z}{\lambda}) - h_2(u, v, z)| = 0, \quad \forall \epsilon > 0, \quad (7.1)$$

$$\sqrt{\lambda} \tilde{g}_2([\lambda u], [\sqrt{\lambda} v], 1 - \frac{z}{\lambda}) \leq C(\bar{h}_2(u, v, z) + \sqrt{\lambda} e^{-zu - c(\lambda u)^{1/3} - c(\sqrt{\lambda} |v|)^{1/2}}), \quad \forall u, z > 0, v \in \mathbb{R}. \quad (7.2)$$

Let  $b(t; k, p)$  denote the binomial distribution with success probability  $p \in (0, 1)$ :

$$b(t; k, p) := \frac{k!}{t!(k-t)!} p^t (1-p)^{k-t}, \quad k = 0, 1, \dots, \quad t = 0, 1, \dots, k. \quad (7.3)$$

We shall need the following version of the Moivre-Laplace theorem (see [15], vol.I, ch.7, §2, Thm.1): *There exists a constant  $C$  such that when  $k \rightarrow \infty$  and  $t \rightarrow \infty$  vary in such a way that*

$$\frac{(t - pk)^3}{k^2} \rightarrow 0, \quad (7.4)$$

then

$$\left| \frac{b(t; k, p)}{\frac{1}{\sqrt{2\pi kp(1-p)}} \exp\{-\frac{(t-kp)^2}{2kp(1-p)}\}} - 1 \right| < \frac{C}{k} + \frac{C|t - pk|^3}{k^2}. \quad (7.5)$$

For  $p(u, v)$  in (4.6), (7.4)-(7.5) imply that there exist  $K_0 > 0$  and  $C > 0$  such that

$$\sup_{u>0, v \in \mathbb{Z}} \left| \frac{p(u, v)}{\sqrt{\frac{2}{\pi u}} e^{-v^2/2u}} - 1 \right| \mathbf{1}(u^2 > K|v|^3, u > K, u \equiv^{\text{mod } 2} v) < \frac{C}{K}, \quad \forall K > K_0. \quad (7.6)$$

Relations (4.5), (7.6) and  $\lim_{\lambda \rightarrow \infty} \sup_{\epsilon < u, z < 1/\epsilon} |(1 - \frac{z}{\lambda})^{[\lambda u]} - e^{-zu}| = 0$  easily imply (7.1).

Consider (7.2). Split  $h_\lambda(u, v, z) := \sqrt{\lambda} \tilde{g}_2([\lambda u], [\sqrt{\lambda} v], 1 - z/\lambda) \leq \sum_{i=1}^3 h_{\lambda i}(u, v, z)$ , where

$$\begin{aligned} h_{\lambda 1}(u, v, z) &:= h_\lambda(u, v, z) \mathbf{1}(\sqrt{\lambda} u^2 > K|v|^3, \lambda u > K), \\ h_{\lambda 2}(u, v, z) &:= h_\lambda(u, v, z) \mathbf{1}(\sqrt{\lambda} u^2 \leq K|v|^3, \lambda u > K), \quad h_{\lambda 3}(u, v, z) := h_\lambda(u, v, z) \mathbf{1}(\lambda u \leq K). \end{aligned}$$

Then, (7.6) together with (4.7) and  $0 \leq 1 - \frac{z}{\lambda} \leq e^{-z/\lambda}$ ,  $0 < z < \lambda$  imply that

$$h_{\lambda 1}(u, v, z) < \frac{C}{\sqrt{u}} e^{-zu - \frac{[\sqrt{\lambda} v]^2}{2\lambda u}} \left(1 + \frac{1}{K}\right), \quad \forall K > K_0, \quad \forall u > 0, v \in \mathbb{R}, 0 < z < \lambda. \quad (7.7)$$

Note that  $\sqrt{\lambda}|v| \geq 2$  implies  $[\sqrt{\lambda} v]^2 \geq (1/4)\lambda v^2$ , while  $\sqrt{\lambda}|v| < 2$  and  $\lambda u > K \geq 1$  imply  $e^{-v^2/8u} > e^{-1/2} = C$ . Hence and from (7.7) we obtain

$$h_{\lambda 1}(u, v, z) < C \bar{h}_2(u, v, z), \quad \forall u > 0, v \in \mathbb{R}, 0 < z < \lambda. \quad (7.8)$$

To estimate  $h_{\lambda 2}$ , we use the well-known Hoeffding's inequality [23]. Let  $b(t; k, p)$  be the binomial distribution in (7.3). Then for any  $\tau > 0$

$$\sum_{|t - kp| > \tau \sqrt{k}} b(t; k, p) \leq 2e^{-2\tau^2}. \quad (7.9)$$

In terms of  $p(u, v)$  of (4.6), inequality (7.9) writes as

$$\sum_{|v| > 2\tau \sqrt{u}} p(u, v) \leq 2e^{-2\tau^2}, \quad \forall \tau > 0. \quad (7.10)$$

We shall also use the following bound

$$p(u, v) \leq 2e^{-v^2/2u}, \quad \forall u, v \in \mathbb{Z}, u \geq 0, |v| \leq u, \quad (7.11)$$

which easily follows from (7.10). Using (7.11), for any  $u > 0$ ,  $v \in \mathbb{R}$ ,  $0 < z < \lambda$ ,  $\lambda > 0$ ,  $K > 0$  we obtain

$$\begin{aligned} h_{\lambda 2}(u, v, z) &< 2\sqrt{\lambda} e^{-zu - \frac{[\sqrt{\lambda} v]^2}{2[\lambda u]}} \mathbf{1}(\sqrt{\lambda} u^2 \leq K|v|^3, \lambda u > K) \\ &\leq C(K) \sqrt{\lambda} \exp \left\{ -zu - (1/8) \max \left( \frac{(\lambda u)^{1/3}}{K^{2/3}}, \frac{(\sqrt{\lambda}|v|)^{1/2}}{K^{1/2}} \right) \right\} \\ &\leq C(K) \sqrt{\lambda} \exp \left\{ -zu - \frac{(\lambda u)^{1/3}}{16K^{2/3}} - \frac{(\sqrt{\lambda}|v|)^{1/2}}{16K^{1/2}} \right\}. \end{aligned} \quad (7.12)$$

Indeed,  $[\sqrt{\lambda}|v|] \geq \sqrt{\lambda}|v| - 1 \geq \frac{\sqrt{\lambda}|v|}{2}$  for  $|v| > 2/\sqrt{\lambda}$  and hence

$$\frac{[\sqrt{\lambda}|v|]^2}{2[\lambda u]} \geq \frac{v^2}{8u} \geq \frac{1}{8} \max \left( \frac{(\lambda u)^{1/3}}{K^{2/3}}, \frac{(\sqrt{\lambda}|v|)^{1/2}}{K^{1/2}} \right) \geq \frac{(\lambda u)^{1/3}}{16K^{2/3}} + \frac{(\sqrt{\lambda}|v|)^{1/2}}{16K^{1/2}} \quad (7.13)$$

holds for  $\sqrt{\lambda}u^2 \leq K|v|^3$ ,  $|v| > 2/\sqrt{\lambda}$ . On the other hand,  $(\sqrt{\lambda}u^2/K)^{1/3} < |v| < 2/\sqrt{\lambda}$  implies  $\lambda u < 2^{3/2}K^{1/2}$  in which case the r.h.s. of (7.13) does not exceed  $(\sqrt{2}/16)(1 + \frac{1}{\sqrt{K}}) =: c(K)$  and (7.12) holds with  $C(K) = e^{c(K)}$ . A similar bound as in (7.12) follows for  $h_{\lambda 3}(u, v, z)$ , using  $h_{\lambda 3}(u, v, z) \leq \sqrt{\lambda}e^{-zu}p([\lambda u], [\sqrt{\lambda}v])\mathbf{1}(\lambda u \leq K) \leq \sqrt{\lambda}e^{-zu}\mathbf{1}(\lambda u \leq K, |[\sqrt{\lambda}v]| \leq K)$ . The desired inequality in (7.2) now follows by combining (7.8) and (7.12) and taking  $K > K_0$  a fixed and sufficiently large number. Lemma 4.1 is proved.

*Proof of Lemma 5.1.* Let us first explain the idea behind the derivation of (1.19). Write  $W_k = (W_{1k}, W_{2k}) \in \mathbb{Z}^2$ . Note  $W_{1k}$  has the binomial distribution with success probability  $1/3$  and, conditioned on  $W_{1k} = t$ ,  $W_{2k}$  is a sum of  $k - t$  Bernoulli r.v.'s taking values  $\pm 1$  with probability  $1/2$ . Hence for  $k \geq t, k - t \geq |s|$  and  $k - t + s$  even,

$$\begin{aligned} p_k(t, s) &= P(W_{1k} = t, W_{2k} = s) = P(W_{k1} = t)P(W_{k2} = s | W_{k1} = t) \\ &= b(t; k, \frac{1}{3})p(k - t, s). \end{aligned} \quad (7.14)$$

Here,  $p(k - t, s) = b((k - t + s)/2; k - t, \frac{1}{2})$  is the same as in (4.6) and  $b(t; k, p)$  is the binomial probability in (7.3). Using (7.14) and the Moivre-Laplace approximation in (7.5), we can write

$$\begin{aligned} \sqrt{\lambda}g_3([\lambda t], [\sqrt{\lambda}s], 1 - \lambda^{-1}z) &= \sqrt{\lambda} \sum_{k=[\lambda t]}^{\infty} (1 - \frac{z}{\lambda})^k p_k([\lambda t], [\sqrt{\lambda}s]) \\ &\sim \frac{3}{2\lambda} \sum_{k=[\lambda t]}^{\infty} e^{-z(k/\lambda)} \sqrt{\frac{\lambda}{4\pi(\frac{k}{\lambda})}} e^{-\frac{(3\lambda t - k)^2}{12\lambda t \frac{k}{\lambda t}}} \frac{1}{\sqrt{(\pi/2)(\frac{k}{\lambda} - t)}} e^{-\frac{(\frac{s}{2})^2}{(1/2)(\frac{k}{\lambda} - t)}} \\ &\sim \frac{3}{2} \int_t^{\infty} e^{-zx} \sqrt{\frac{\lambda}{4\pi x}} e^{-\frac{\lambda(3t-x)^2}{4x}} \frac{1}{\sqrt{(\pi/2)(x-t)}} e^{-\frac{(\frac{s}{2})^2}{(1/2)(x-t)}} dx \\ &\sim \frac{3}{2} \int_t^{\infty} e^{-zx} \sqrt{\frac{\lambda}{12\pi t}} e^{-\frac{\lambda(3t-x)^2}{12t}} \frac{1}{\sqrt{(\pi/2)(x-t)}} e^{-\frac{(\frac{s}{2})^2}{(1/2)(x-t)}} dx \\ &\sim \frac{3}{2\sqrt{\pi t}} e^{-3zt - \frac{s^2}{4t}} = h_3(t, s, z). \end{aligned} \quad (7.15)$$

Here, factor  $\frac{1}{2}$  in front of the second sum comes from the fact that  $p_k(t, s) = 0$  for  $k - t \not\equiv s \pmod{2}$ , while factor  $\sqrt{\frac{\lambda}{4\pi(\frac{k}{\lambda})}} \exp\{-\frac{(3\lambda t - k)^2}{(12\lambda t)(\frac{k}{\lambda t})}\} \sim \sqrt{\frac{\lambda}{12\pi t}} \exp\{-\frac{\lambda(3t-x)^2}{12t}\}$  behaves as a delta-function in a neighborhood of  $k = 3\lambda t$  or  $x = 3t$ , resulting in the asymptotic formula (7.15).

Let us turn to the rigorous proof of (7.15) and Lemma 5.1. Split

$$h_{\lambda}(t, s, z) := \sqrt{\lambda}g_3([\lambda t], [\sqrt{\lambda}s], 1 - \lambda^{-1}z) = \sum_{i=0}^5 h_{\lambda i}(t, s, z), \quad (7.16)$$

where

$$\begin{aligned} h_{\lambda 0}(t, s, z) &:= \sqrt{\lambda} \sum_{k=[\lambda t]}^{\infty} (1 - \frac{z}{\lambda})^k p_k([\lambda t], [\sqrt{\lambda}s]) \mathbf{1}(\lambda t \leq K), \\ h_{\lambda 1}(t, s, z) &:= \sqrt{\lambda} \sum_{k=[\lambda t]}^{\infty} (1 - \frac{z}{\lambda})^k p_k([\lambda t], [\sqrt{\lambda}s]) \mathbf{1}(K|3\lambda t - k|^3 \geq k^2, \lambda t > K), \\ h_{\lambda 2}(t, s, z) &:= \sqrt{\lambda} \sum_{k=[\lambda t]}^{\infty} (1 - \frac{z}{\lambda})^k b([\lambda t]; k, \frac{1}{3}) \{p(k - [\lambda t], [\sqrt{\lambda}s]) - \bar{p}([2\lambda t], [\sqrt{\lambda}s])\} \mathbf{1}(K|3\lambda t - k|^3 < k^2, \lambda t > K), \\ h_{\lambda 3}(t, s, z) &:= \sqrt{\lambda} \bar{p}([2\lambda t], [\sqrt{\lambda}s]) \sum_{k=[\lambda t]}^{\infty} \left\{ (1 - \frac{z}{\lambda})^k - (1 - \frac{z}{\lambda})^{3\lambda t} \right\} b([\lambda t]; k, \frac{1}{3}) \mathbf{1}(K|3\lambda t - k|^3 < k^2, \lambda t > K), \\ h_{\lambda 4}(t, s, z) &:= \sqrt{\lambda} \bar{p}([2\lambda t], [\sqrt{\lambda}s]) (1 - \frac{z}{\lambda})^{3\lambda t} (V_{\lambda}(t) - 3), \\ h_{\lambda 5}(t, s, z) &:= 3\sqrt{\lambda} \bar{p}([2\lambda t], [\sqrt{\lambda}s]) (1 - \frac{z}{\lambda})^{3\lambda t}, \end{aligned}$$

and where  $\bar{p}(t, s) := (p(t, s) + p(t, s + 1))/2$ ,  $t \in \mathbb{N}$ ,  $s \in \mathbb{Z}$  and

$$V_\lambda(t) := \sum_{k=[\lambda t]}^{\infty} b([\lambda t]; k, \frac{1}{3}) \mathbf{1}(K|3\lambda t - k|^3 < k^2, \lambda t > K).$$

Here,  $h_{\lambda 5}$  is the main term and  $h_{\lambda i}, i = 0, 1, \dots, 4$  are remainder terms. In particular, we shall prove that

$$\lim_{K \rightarrow \infty} \limsup_{\lambda \rightarrow \infty} \sup_{\epsilon < t, |s|, z < 1/\epsilon} |h_{\lambda i}(t, s, z)| = 0, \quad \forall i = 0, 1, 2, 3, 4, \quad \forall \epsilon > 0. \quad (7.17)$$

Relations (7.17) are used to prove (1.19). The proof of (5.5) also uses the decomposition (7.16), with  $K > 0$  a fixed large number.

Step 1 (estimation of  $h_{\lambda 5}$ ). For any  $\epsilon > 0$ ,

$$\lim_{\lambda \rightarrow \infty} \sup_{\epsilon < t, |s|, z < 1/\epsilon} |h_{\lambda 5}(t, s, z) - h_3(t, s, z)| = 0. \quad (7.18)$$

Moreover, there exist constants  $C, c > 0$  such that for all sufficiently large  $\lambda$  and any  $(t, s, z) \in \mathbb{R}^3$ ,  $t > 0$ ,  $s \in \mathbb{R}$ ,  $0 < z < \lambda$  the following inequality holds

$$|h_{\lambda 5}(t, s, z)| < C(\bar{h}_3(t, s, z) + \sqrt{\lambda} e^{-zt - c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}). \quad (7.19)$$

Relations (7.18) and (7.19) follow as in the proof of Lemma 4.1, (7.1) and (7.2), respectively.

Step 2 (estimation of  $h_{\lambda 4}$ ). Let us show (7.17) for  $i = 4$  and that there exist constants  $C, c > 0$  such that for all sufficiently large  $\lambda$  and any  $(t, s, z) \in \mathbb{R}^3$ ,  $t > 0$ ,  $s \in \mathbb{R}$ ,  $0 < z < \lambda$  the following inequality holds:

$$|h_{\lambda 4}(t, s, z)| < C(\bar{h}_3(t, s, z) + \sqrt{\lambda} e^{-zt - c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}). \quad (7.20)$$

Indeed,  $|h_{\lambda 4}(t, s, z)| \leq C h_{\lambda 5}(t, s, z) |V_\lambda(t) - 3|$ . Therefore the above facts ((7.17) for  $i = 4$  and (7.20)) follow from Step 1 and the following bound: There exist  $C, c > 0$  and  $K_0 > 0$  such that

$$|V_\lambda(t) - 3| < C(K^{-1/3} + e^{-c(\sqrt{\lambda}t/K)^{2/3}}), \quad \forall \lambda > 0, t > 0, \lambda t > K, K > K_0. \quad (7.21)$$

To show (7.21) we use the Moivre-Laplace approximation in (7.5). Accordingly,  $V_\lambda(t) = V_{\lambda 1}(t) + V_{\lambda 2}(t)$ , where

$$V_{\lambda 1}(t) := \frac{3}{2\sqrt{\pi}} \sum_{k=[\lambda t]}^{\infty} \frac{1}{\sqrt{k}} e^{-(3[\lambda t] - k)^2/4k} \mathbf{1}(K|3\lambda t - k|^3 < k^2, \lambda t > K)$$

and where  $V_{\lambda 2}(t)$  satisfies

$$|V_{\lambda 2}(t)| < (C/K) V_{\lambda 1}(t)$$

for all  $\lambda > 0$ ,  $t > 0$ ,  $\lambda t > K$ ,  $K > K_0$  and some  $C > 0$  and  $K_0 > 0$  independent of  $\lambda, t$ , and  $K$ . Hence, it suffices to prove (7.21) for  $V_{\lambda 1}(t)$  instead of  $V_\lambda(t)$ .

Let  $\mathcal{D}_K(\tau) := \{k \in \mathbb{N} : K|3\tau - k|^3 < k^2\}$ ,  $\tau > 0$ . There exist  $C > 0$  and  $\tau_0 > 0$  such that  $k \in \mathcal{D}_K(\tau)$  implies

$$|k - 3\tau| < C\tau^{2/3}/K^{1/3} \quad \text{and} \quad 2\tau < k < 4\tau, \quad \forall \tau > \tau_0. \quad (7.22)$$

Indeed, let  $k \leq 3\tau$ . Then  $|k - 3\tau| < k^{2/3}/K^{1/3} \leq 3^{2/3}\tau^{2/3}/K^{1/3}$  and the first inequality in (7.22) holds with  $C = 3^{2/3}$ . Next, let  $k > 3\tau$ . Then  $k^{2/3}/K^{1/3} < k/4$  for  $\tau > \tau_0$  and some  $\tau_0 > 0$  and hence  $k - 3\tau < k/4$  implying  $k < 4\tau$ . In turn this implies  $|k - 3\tau| < (4\tau)^{2/3}/K^{1/3}$  and (7.22) holds with  $C = 4^{2/3}$ .

Consider  $\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{3\lambda t}} = \frac{1}{\sqrt{3\lambda t}} \left( \frac{1}{\sqrt{1 + \frac{k-3\lambda t}{3\lambda t}}} - 1 \right)$ . Using  $|1 - \frac{1}{\sqrt{1+x}}| \leq |x|$  and  $|1 - \frac{1}{1+x}| \leq 2|x|$  for  $|x| \leq 1/2$  we obtain

$$\left| \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{3\lambda t}} \right| \leq \frac{C}{\sqrt{\lambda t}} \frac{1}{K^{1/3}(\lambda t)^{1/3}} < \frac{C}{\sqrt{\lambda t} K^{2/3}}, \quad \left| \frac{1}{k} - \frac{1}{3\lambda t} \right| < \frac{C}{(\lambda t)^{4/3} K^{1/3}}, \quad (7.23)$$



for some constant  $C < \infty$  and all  $|k - 3\lambda t| < C(\lambda t)^{2/3}/K^{1/3}$ ,  $\lambda t > K > K_0$  and  $K_0 > 0$  large enough. From (7.23) and (7.22), for the above values of  $k, \lambda, t, K$  we obtain

$$\left| \frac{1}{\sqrt{k}} e^{-(3[\lambda t] - k)^2/4k} - \frac{1}{\sqrt{3\lambda t}} e^{-(3[\lambda t] - k)^2/12\lambda t} \right| < \frac{C}{K^{1/3}} \frac{1}{\sqrt{\lambda t}} e^{-(3[\lambda t] - k)^2/12\lambda t}.$$

Hence,  $|V_{\lambda 1}(t) - 3| \leq |U_{\lambda 1}(t) - 3| + U_{\lambda 2}(t) + \left(\frac{C}{K^{1/3}}\right)U_{\lambda 3}(t)$ , where

$$\begin{aligned} U_{\lambda 1}(t) &:= \frac{3}{2\sqrt{3\pi\lambda t}} \sum_{k=[\lambda t]}^{\infty} e^{-(3[\lambda t] - k)^2/12\lambda t} \mathbf{1}(\lambda t > K), \\ U_{\lambda 2}(t) &:= \frac{3}{2\sqrt{3\pi\lambda t}} \sum_{k=[\lambda t]}^{\infty} e^{-(3[\lambda t] - k)^2/12\lambda t} \mathbf{1}(K|3\lambda t - k|^3 \geq k^2, \lambda t > K), \\ U_{\lambda 3}(t) &:= \frac{1}{\sqrt{\lambda t}} \sum_{k \in \mathbb{Z}} e^{-k^2/12\lambda t} \mathbf{1}(\lambda t > K). \end{aligned}$$

It is easy to show that  $U_{\lambda 3}(t) < C$  and  $|U_{\lambda 1}(t) - 3| = |U_{\lambda 1}(t) - \frac{3}{2\sqrt{3\pi}} \int_{\mathbb{R}} e^{-x^2/12} dx| < C/\sqrt{\lambda t} < C/K^{1/2}$  uniformly in  $\lambda > 0, t > 0, K > K_0$ . Next, with  $j = k - 3[\lambda t]$  and using the fact that  $k^2 = (j + 3[\lambda t])^2 \geq [\lambda t]^2 \geq (\lambda t)^2/2$

$$\begin{aligned} |U_{\lambda 2}(t)| &\leq \frac{C}{\sqrt{\lambda t}} \sum_{j=-2[\lambda t]}^{\infty} e^{-(\frac{j}{\sqrt{\lambda t}})^2/12} \mathbf{1}\left(K \left|\frac{j}{\sqrt{\lambda t}}\right|^3 \geq \frac{(\lambda t)^2}{2(\lambda t)^{3/2}}\right) \\ &\leq C \int \mathbf{1}(K|x|^3 > \sqrt{\lambda t}/2) e^{-x^2} dx \leq C e^{-c(\sqrt{\lambda t}/K)^{2/3}}. \end{aligned}$$

This proves (7.21) and hence (7.20), too.

Step 3 (estimation of  $h_{\lambda 3}$ ). First we estimate the difference inside the curly brackets. There exist  $C, K_0, \tau_0 > 0$  such that  $k \in \mathcal{D}_K(\tau)$ ,  $K > K_0$ ,  $\tau > \tau_0$  imply

$$|a^k - a^{3\tau}| \leq C a^{2\tau} \frac{\tau^{2/3}}{K^{1/3}} |1 - a|, \quad \forall a \in [0, 1]. \quad (7.24)$$

Indeed, let  $k \leq 3\tau$ . Using (7.22) and  $1 - a^\tau \leq (1 + \tau)(1 - a)$ ,  $\forall \tau \geq 0, \forall a \in [0, 1]$ , for sufficiently large  $\tau > K$  we obtain  $|a^k - a^{3\tau}| \leq a^k |a^{3\tau - k} - 1| \leq a^{2\tau} |3\tau - k + 1| |1 - a| \leq C a^{2\tau} \frac{\tau^{2/3} + 1}{K^{1/3}} |1 - a| < \frac{C}{K^{1/3}} a^{2\tau} \tau^{2/3} |1 - a|$ . The case  $k > 3\tau$  in (7.24) follows analogously. Using (7.24) and (7.21), together with the inequality  $ze^{-2z} < Ce^{-z}$ ,  $z > 0$ , we obtain

$$\begin{aligned} |h_{\lambda 3}(t, s, z)| &< \sqrt{\lambda} \bar{p}([2\lambda t], [\sqrt{\lambda} s]) \left(1 - \frac{z}{\lambda}\right)^{2\lambda t} \frac{(\lambda t)^{2/3} (z/\lambda)}{K^{1/3}} V_{\lambda}(t) \\ &< C \sqrt{\lambda} \bar{p}([2\lambda t], [\sqrt{\lambda} s]) e^{-2zt} \frac{(\lambda t)^{2/3} (zt)}{\lambda t} \\ &< \frac{C}{(\lambda t)^{1/3}} \sqrt{\lambda} \bar{p}([2\lambda t], [\sqrt{\lambda} s]) e^{-zt}. \end{aligned}$$

Therefore as in Step 2 we obtain the convergence in (7.17) for  $i = 3$  together with the bound

$$|h_{\lambda 3}(t, s, z)| < C(\bar{h}_3(t, s, z) + \sqrt{\lambda} e^{-zt - c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}). \quad (7.25)$$

Step 4 (estimation of  $h_{\lambda 2}$ ). First we estimate the difference inside the curly brackets. There exist  $C > 0$  and  $K_0 > 0$  such that for any  $\lambda, t, s, k, K$  satisfying

$$\lambda > 0, \quad t > 0, \quad s \in \mathbb{R}, \quad k \in \mathbb{N}, \quad K > K_0, \quad \lambda t > K, \quad K|k - 3\lambda t|^3 \leq k^2, \quad \lambda^{1/2} t^2 > K|s|^3, \quad (7.26)$$

the following inequality holds

$$|\bar{p}(k - [\lambda t], [\sqrt{\lambda} s]) - \bar{p}([2\lambda t], [\sqrt{\lambda} s])| \leq \frac{C}{(\lambda t)^{1/2} K^{2/3}} e^{-s^2/10t}. \quad (7.27)$$

In the proof of (7.27), below, assume that  $k - [\lambda t] \stackrel{\text{mod } 2}{=} [\sqrt{\lambda} s]$ ,  $[2\lambda t] \stackrel{\text{mod } 2}{=} [\sqrt{\lambda} s]$ ; the remaining cases can be discussed analogously. Using the Moivre-Laplace formula (7.6) we have that

$$\begin{aligned} & p(k - [\lambda t], [\sqrt{\lambda} s]) - p([2\lambda t], [\sqrt{\lambda} s]) \\ &= \frac{2}{\sqrt{2\pi(k - [\lambda t])}} e^{-\frac{[\sqrt{\lambda} s]^2}{2(k - [\lambda t])}} \left(1 + O\left(\frac{1}{K}\right)\right) - \frac{2}{\sqrt{2\pi[2\lambda t]}} e^{-\frac{[\sqrt{\lambda} s]^2}{2[2\lambda t]}} \left(1 + O\left(\frac{1}{K}\right)\right). \end{aligned}$$

As in (7.23),

$$\left| \frac{1}{\sqrt{k - [\lambda t]}} - \frac{1}{\sqrt{[2\lambda t]}} \right| \leq \frac{C}{(\lambda t)^{1/2} K^{2/3}}, \quad \left| \frac{1}{k - [\lambda t]} - \frac{1}{[2\lambda t]} \right| \leq \frac{C}{(\lambda t)^{4/3} K^{1/3}}.$$

Hence it easily follows  $e^{-\frac{[\sqrt{\lambda} s]^2}{2(k - [\lambda t])}} < C e^{-\frac{s^2}{10t}}$ , where the arguments satisfy (7.26). The above facts imply (7.27). Using (7.27) for  $s$  satisfying (7.26) we can write

$$\begin{aligned} |h_{\lambda 2}(t, s, z)| &< \frac{C}{t^{1/2} K^{2/3}} e^{-s^2/10t} \sum_{k=[\lambda t]}^{\infty} \left(1 - \frac{z}{\lambda}\right)^k b([\lambda t]; k, \frac{1}{3}) \mathbf{1}(K|3\lambda t - k|^3 < k^2, \lambda t > K) \\ &< \frac{C}{K^{2/3}} \bar{h}_3(t, s, z) V_{\lambda}(t) < \frac{C}{K^{2/3}} \bar{h}_3(t, s, z), \end{aligned} \quad (7.28)$$

see (7.21).

Next we will evaluate  $h_{\lambda 2}(t, s, z)$  for  $\lambda^{1/2} t^2 < K|s|^3$  (and  $\lambda, t, k, K$  satisfying (7.26)). Using the inequality in (7.11), the bound  $k < 4\lambda t$ , see (7.22), and arguing as in (7.13) we have that

$$\frac{[\sqrt{\lambda} s]^2}{2(k - [\lambda t])} > \frac{s^2}{6t} > \max \left\{ \frac{(\sqrt{\lambda} s)^{1/2}}{6K^{1/2}}, \frac{(\lambda t)^{1/3}}{6K^{2/3}} \right\}$$

and hence

$$p([\lambda t], [\sqrt{\lambda} s]) \leq C e^{-c(K)(\lambda t)^{1/3} - c(K)(\sqrt{\lambda}|s|)^{1/2}},$$

where  $c(K) > 0$  depends only on  $K$ . Therefore,

$$\begin{aligned} |h_{\lambda 2}(t, s, z)| &< C \sqrt{\lambda} e^{-2zt - c(K)(\lambda t)^{1/3} - c(K)(\sqrt{\lambda}|s|)^{1/2}} V_{\lambda}(t) \\ &< C \sqrt{\lambda} e^{-2zt - c(K)(\lambda t)^{1/3} - c(K)(\sqrt{\lambda}|s|)^{1/2}}. \end{aligned} \quad (7.29)$$

The resulting bound

$$|h_{\lambda 2}(t, s, z)| < C (\bar{h}_3(t, s, z) + \sqrt{\lambda} e^{-zt - c(K)(\lambda t)^{1/3} - c(K)(\sqrt{\lambda}|s|)^{1/2}}) \quad (7.30)$$

follows from (7.28) and (7.29) by taking  $K > K_0$  sufficiently large but fixed.

Step 5 (estimation of  $h_{\lambda 1}$ ). From (7.9), we have  $b([\lambda t]; k, 1/3) \leq 2e^{-(2/9)|3[\lambda t] - k|^2/k}$ . Using this and a similar inequality (7.11) for  $p(k - [\lambda t], [\sqrt{\lambda} s])$  we see that

$$\begin{aligned} |h_{\lambda 1}(t, s, z)| &< C \sqrt{\lambda} e^{-zt} \sum_{k=[\lambda t]}^{\infty} e^{-(2/9)\frac{|3[\lambda t] - k|^2}{k}} e^{-(1/2)\frac{[\sqrt{\lambda} s]^2}{k - [\lambda t]}} \mathbf{1}(K|3\lambda t - k|^3 \geq k^2, \lambda t > K) \\ &< C \sqrt{\lambda} e^{-zt} \sum_{k \geq \lambda t} e^{-ck^{1/3} - c\frac{(\sqrt{\lambda} s)^2}{k - \lambda t}} \end{aligned}$$

for some positive constant  $c > 0$  depending on  $K$ . Split the sum  $\sum_{k \geq \lambda t} \dots = \sum_{\lambda t < k \leq \lambda t + (\sqrt{\lambda}|s|)^{3/2}} \dots + \sum_{k > \lambda t + (\sqrt{\lambda}|s|)^{3/2}} \dots =: \Sigma_1 + \Sigma_2$ . Then  $\Sigma_1 < C e^{-c(\sqrt{\lambda}|s|)^{1/2}} \sum_{k \geq \lambda t} e^{-ck^{1/3}} < C e^{-c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}$  and

$$\Sigma_2 < C \sum_{k \geq \lambda t + (\sqrt{\lambda}|s|)^{3/2}} e^{-ck^{1/3}} < C e^{-c(\lambda t + (\sqrt{\lambda}|s|)^{3/2})^{1/3}} < C e^{-c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}.$$

By taking  $K > K_0$  sufficiently large but fixed the above calculations lead to the bound

$$|h_{\lambda 1}(t, s, z)| < C \sqrt{\lambda} e^{-zt - c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}. \quad (7.31)$$

Step 6 (estimation of  $h_{\lambda 0}$ ). Similarly as in Step 5 we obtain

$$|h_{\lambda 0}(t, s, z)| < C\sqrt{\lambda}e^{-zt-c(\lambda t)^{1/3}-c(\sqrt{\lambda}|s|)^{1/2}}. \quad (7.32)$$

The proof of Lemma 5.1 follows from Steps 1 - 6.

*Proof of Lemma 6.1.* Let  $W_k = (W_{1k}, W_{2k}) \in \mathbb{Z}^2$  and

$$\widetilde{W}_{1k} := W_{1k} + W_{2k}, \quad \widetilde{W}_{2k} := W_{1k} - W_{2k}.$$

Then  $\widetilde{W}_k = (\widetilde{W}_{1k}, \widetilde{W}_{2k})$ ,  $k = 0, 1, \dots$  is a random walk on the even lattice

$$\widetilde{\mathbb{Z}}^2 := \{(u, v) \in \mathbb{Z}^2 : u + v \text{ is even}\} = \{(u, v) \in \mathbb{Z}^2 : u \equiv v \pmod{2}\} \quad (7.33)$$

with one-step transition probabilities

$$P(\widetilde{W}_1 = (i, j) | \widetilde{W}_0 = (0, 0)) = 1/4, \quad i, j = \pm 1.$$

Note that  $\{\widetilde{W}_{1k}\}$  and  $\{\widetilde{W}_{2k}\}$  are independent symmetric random walks on  $\mathbb{Z}$  and therefore

$$\tilde{p}_k(u, v) := P(\widetilde{W}_k = (u, v) | \widetilde{W}_0 = (0, 0)) = p(k, u)p(k, v), \quad (u, v) \in \widetilde{\mathbb{Z}}^2, \quad k = 0, 1, \dots,$$

where  $p(u, v)$  is the  $u$ -th step transition probability for the symmetric random walk on  $\mathbb{Z}$  as given in (4.6). The above facts imply the following factorization property:

$$p_k(t, s) = \tilde{p}_k(t + s, t - s) = p(k, t + s)p(k, t - s), \quad t, s \in \mathbb{Z}, \quad k = 0, 1, \dots \quad (7.34)$$

In particular,  $p_k(t, s) = 0$  if  $k \not\equiv t + s \pmod{2}$ . Split  $g_4([\lambda t], [\lambda s], 1 - \frac{z}{\lambda^2}) = \sum_{i=1}^3 \gamma_{\lambda i}(t, s, z)$ , where

$$\gamma_{\lambda i}(t, s, z) := \lambda^2 \int_0^\infty \left(1 - \frac{z}{\lambda^2}\right)^{[\lambda^2 x]} p_{[\lambda^2 x]}([\lambda t], [\lambda s]) \mathbf{1}(x \in I_{\lambda i}(t, s)) dx, \quad i = 1, 2, 3,$$

and where

$$\begin{aligned} I_{\lambda 1}(t, s) &:= \left\{x > 0 : \lambda x^2 > K(|t|^3 + |s|^3), \lambda^2 x > K, [\lambda^2 x] \equiv [\lambda t] + [\lambda s] \pmod{2}\right\}, \\ I_{\lambda 2}(t, s) &:= \left\{x > 0 : \lambda x^2 \leq K(|t|^3 + |s|^3), \lambda^2 x > K, [\lambda^2 x] \equiv [\lambda t] + [\lambda s] \pmod{2}\right\}, \\ I_{\lambda 3}(t, s) &:= \left\{x > 0 : \lambda^2 x \leq K, [\lambda^2 x] \equiv [\lambda t] + [\lambda s] \pmod{2}\right\} \end{aligned}$$

satisfy  $\bigcup_{i=1}^3 I_{\lambda i}(t, s) = I_{\lambda 0}(t, s) := \left\{x > 0 : [\lambda^2 x] \equiv [\lambda t] + [\lambda s] \pmod{2}\right\}$ . Also split

$$h_4(t, s, z) = \pi^{-1} \int_0^\infty x^{-1} e^{-zx - \frac{t^2 + s^2}{x}} dx = \sum_{i=0}^3 h_{\lambda i}(t, s, z),$$

where

$$\begin{aligned} h_{\lambda 0}(t, s, z) &:= \pi^{-1} \int_0^\infty x^{-1} e^{-zx - \frac{t^2 + s^2}{x}} (1 - 2\mathbf{1}(x \in I_{\lambda 0}(t, s))) dx, \\ h_{\lambda i}(t, s, z) &:= 2\pi^{-1} \int_0^\infty x^{-1} e^{-zx - \frac{t^2 + s^2}{x}} \mathbf{1}(x \in I_{\lambda i}(t, s)) dx, \quad i = 1, 2, 3. \end{aligned}$$

We shall prove below

$$\lim_{\lambda, K \rightarrow \infty} \sup_{\epsilon < |t| + |s| < 1/\epsilon, \epsilon < z < 1/\epsilon} (|\gamma_{\lambda 1}(t, s, z) - h_{\lambda 1}(t, s, z)| + |h_{\lambda 0}(t, s, z)|) = 0, \quad \forall \epsilon > 0, \quad (7.35)$$

and that for any sufficiently large  $K > K_0$  there exist  $c(K), C(K) < \infty$  independent of  $t, s, z, \lambda$  and such that for any  $(t, s) \in \mathbb{R}_0^2$ ,  $0 < z < \lambda^2$  the following inequalities hold:

$$\gamma_{\lambda 1}(t, s, z) + |h_{\lambda 0}(t, s, z)| \leq C(K)h_4(t, s, z), \quad (7.36)$$

$$\gamma_{\lambda i}(t, s, z) + h_{\lambda i}(t, s, z) \leq C(K)e^{-c(K)(|\lambda t|^{1/2} + |\lambda s|^{1/2})}, \quad i = 2, 3. \quad (7.37)$$

Relations (7.36)-(7.37) imply statement (6.5) of the lemma. Statement (6.4) follows from  $|g_4([\lambda t], [\lambda s], 1 - \frac{z}{\lambda^2}) - h_4(t, s, z)| \leq |h_{\lambda 0}(t, s, z)| + |\gamma_{\lambda 1}(t, s, z) - h_{\lambda 1}(t, s, z)| + \sum_{i=2}^3 (\gamma_{\lambda i}(t, s, z) + h_{\lambda i}(t, s, z))$  and using (7.35) and the bounds in (7.36)-(7.37).

Let us prove (7.36). Clearly,  $|h_{\lambda 0}(t, s, z)| \leq 2h_4(t, s, z)$  by the definition of  $h_{\lambda 0}$  so that we need to estimate  $\gamma_{\lambda 1}$  only. Note (7.6) and (7.34) imply

$$\sup_{x, t, s} \left| \frac{p_{[\lambda^2 x]}([\lambda t], [\lambda s])}{\frac{2}{\pi[\lambda^2 x]}e^{-\frac{[\lambda t]^2 + [\lambda s]^2}{[\lambda^2 x]}}} - 1 \right| \mathbf{1}(x \in I_{\lambda 1}(t, s)) < \frac{C}{K}, \quad \forall K > K_0. \quad (7.38)$$

We also need the bound

$$\sup_{x, t, s} \left| \frac{\frac{2}{\pi[\lambda^2 x]}e^{-\frac{[\lambda t]^2 + [\lambda s]^2}{[\lambda^2 x]}}}{\frac{2}{\pi\lambda^2 x}e^{-\frac{t^2 + s^2}{x}}} - 1 \right| \mathbf{1}(x \in I_{\lambda 1}(t, s)) < \frac{C}{K^{2/3}}, \quad \forall K > K_0. \quad (7.39)$$

which follows from  $|\frac{\lambda^2 x}{[\lambda^2 x]} - 1| < C_1/K$ ,  $|\frac{t^2 + s^2}{x} - \frac{[\lambda t]^2 + [\lambda s]^2}{[\lambda^2 x]}| < C_2/K^{2/3}$  for  $x \in I_{\lambda 1}(t, s)$ , with  $C_1, C_2$  independent of  $x, t, s, \lambda, K$ . From (7.38) and (7.39) we obtain

$$\chi(\lambda, K) := \sup_{x, t, s} \left| \frac{p_{[\lambda^2 x]}([\lambda t], [\lambda s])}{\frac{2}{\pi\lambda^2 x}e^{-\frac{t^2 + s^2}{x}}} - 1 \right| \mathbf{1}(x \in I_{\lambda 1}(t, s)) < \frac{C}{K^{2/3}}, \quad \forall K > K_0. \quad (7.40)$$

Using (7.40) and  $(1 - \frac{z}{\lambda^2})^{[\lambda^2 x]} \leq e^{z/\lambda^2 - z[\lambda^2 x]/\lambda^2} \leq Ce^{-zx}$ ,  $0 < z < \lambda^2$  we obtain

$$\begin{aligned} \gamma_{\lambda 1}(t, s, z) &\leq C\lambda^2 \int_0^\infty e^{-zx} \frac{2}{\pi\lambda^2 x} e^{-\frac{t^2 + s^2}{x}} (1 + \chi(\lambda, K)) \mathbf{1}(x \in I_{\lambda 1}(t, s)) dx \\ &\leq Ch_{\lambda 1}(t, s, z) \leq Ch_4(t, s, z), \quad K > K_0, \end{aligned}$$

proving (7.36), with  $C(K)$  independent of  $K > K_0$ . Similarly using (7.40) we obtain

$$\begin{aligned} |\gamma_{\lambda 1}(t, s, z) - h_{\lambda 1}(t, s, z)| &\leq \left| \int_0^\infty (1 - \frac{z}{\lambda^2})^{[\lambda^2 x]} \left\{ \lambda^2 p_{[\lambda^2 x]}([\lambda t], [\lambda s]) - \frac{2}{\pi x} e^{-\frac{t^2 + s^2}{x}} \right\} \mathbf{1}(x \in I_{\lambda 1}(t, s)) dx \right| \\ &\quad + 2 \left| \int_0^\infty \left\{ (1 - \frac{z}{\lambda^2})^{[\lambda^2 x]} - e^{-zx} \right\} \pi^{-1} x^{-1} e^{-\frac{t^2 + s^2}{x}} \mathbf{1}(x \in I_{\lambda 1}(t, s)) dx \right| \\ &\leq C\chi(\lambda, K)h_4(t, s, z) + C \int_0^\infty \theta_\lambda(z, x) x^{-1} e^{-\frac{t^2 + s^2}{x}} dx \end{aligned}$$

where  $\theta_\lambda(z, x) := |(1 - \frac{z}{\lambda^2})^{[\lambda^2 x]} - e^{-zx}| \rightarrow 0$  ( $\lambda \rightarrow \infty$ ) for any  $z > 0, x > 0$  fixed, and  $|\theta_\lambda(z, x)| \leq Ce^{-xz}$  for any  $x, z, \lambda > 0$ ; see above. Therefore by the dominated convergence theorem,  $\int_0^\infty \theta_\lambda(z, x) x^{-1} e^{-\frac{t^2 + s^2}{x}} dx \rightarrow 0$  as  $\lambda \rightarrow \infty$  and the last convergence is uniform in  $\epsilon < |t| + |s| < 1/\epsilon$ ,  $\epsilon < z < 1/\epsilon$  for any given  $\epsilon > 0$ . Together with (7.40) this proves (7.35) for the difference  $|\gamma_{\lambda 1} - h_{\lambda 1}|$ . Relation (7.35) for  $|h_{\lambda 0}|$  follows by the mean value theorem, implying  $|x^{-1}e^{-zx - \frac{t^2 + s^2}{x}} - y^{-1}e^{-zy - \frac{t^2 + s^2}{y}}| \leq C(\epsilon)|x - y|x^{-1}e^{-zx - \frac{t^2 + s^2}{x}}(1 + x^{-2})$  for  $0 < x < y$ ,  $0 < z < 1/\epsilon$ ,  $|t| + |s| < 1/\epsilon$ . Therefore,  $\sup_{\epsilon < |t| + |s| < 1/\epsilon, \epsilon < z < 1/\epsilon} |h_{\lambda 0}(t, s, z)| \leq C/\lambda^2 = o(1)$ , where  $C := \sup_{\epsilon < |t| + |s| < 1/\epsilon, z > \epsilon} \int_0^\infty x^{-1}e^{-zx - \frac{t^2 + s^2}{x}}(1 + x^{-2})dx < \infty$ .

It remains to prove (7.37). Note  $\gamma_{\lambda 2}(t, s, z) \leq \bar{\gamma}_2([\lambda t], [\lambda s])$ ,  $0 < z < \lambda^2$ , where  $\bar{\gamma}_2(t, s) := \sum p_k(t, s) \mathbf{1}(K < k < \sqrt{K(|t|^3 + |s|^3)})$ ,  $t, s \in \mathbb{Z}$ . Note  $K < k < \sqrt{K(|t|^3 + |s|^3)}$  implies

$$\frac{(|t + s| + |t - s|)^4}{k^2} \geq \frac{(t + s)^4 + (t - s)^4}{K(|t|^3 + |s|^3)} \geq \frac{2(t^4 + s^4)}{K(|t|^3 + |s|^3)} \geq \frac{1}{4K}(|t|^{1/2} + |s|^{1/2})^2.$$

Hence and using (7.11) we obtain

$$\begin{aligned}\bar{\gamma}_2(t, s) &\leq \sum_{K < k < \sqrt{K(|t|^3 + |s|^3)}} p(k, t + s)p(k, t - s) \\ &\leq 4 \sum_{K < k < \sqrt{K(|t|^3 + |s|^3)}} \exp\left\{-\frac{|t|^{1/2} + |s|^{1/2}}{4\sqrt{K}}\right\} < C(K) e^{-c(K)(|t|^{1/2} + |s|^{1/2})},\end{aligned}\quad (7.41)$$

where constants  $C(K) > 0$ ,  $c(K) > 0$  depend only on  $K < \infty$ . This proves (7.37) for  $\gamma_{\lambda 2}$ . The last bound in (7.41) holds for  $\bar{\gamma}_3(t, s) := \sum_{k=0}^K p(k, t + s)p(k, t - s) \leq (K + 1)\mathbf{1}(|t + s| \leq K, |t - s| \leq K)$ , too, dominating  $\gamma_{\lambda 3}(t, s, z) \leq \bar{\gamma}_3([\lambda t], [\lambda s])$ ,  $0 < z < \lambda^2$ . The remaining bounds in (7.37) follow easily. Lemma 6.1 is proved.

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